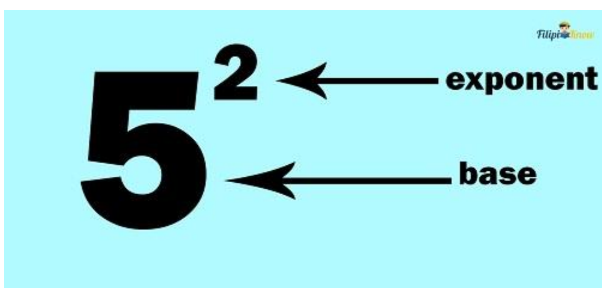
The discovery of exponents revolutionized the way we compute extremely large and small values. Exponents help us understand the concepts of compound interest, population growth, bacterial growth, and radioactive decay. They also serve as a tool to express distances between astronomical bodies, describe computer memory, and express some scientific scales.

In this reviewer, we are going to discuss exponents and the laws that must be applied to perform mathematical operations with them.

Review on Exponents.

What is an exponent?

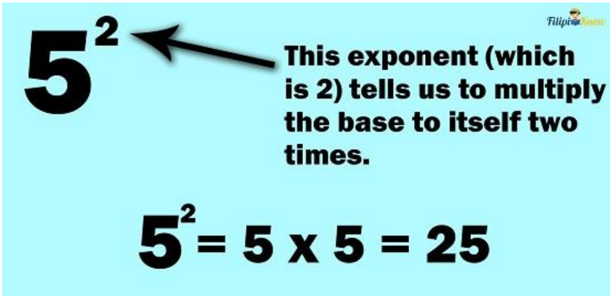
An exponent is a small number that is written on the upper right of another number (or variable) which is called the base. It tells you that the base is raised to the power of the exponent.



For example, in 5^2, the small number (which is 2) written above and on the right of 5 is the exponent. Meanwhile, the number that is written much larger (which is 5) is the base. The exponent tells us that the base, which is 5, is raised to the power of 2.

Computing Whole Numbers with Exponents.

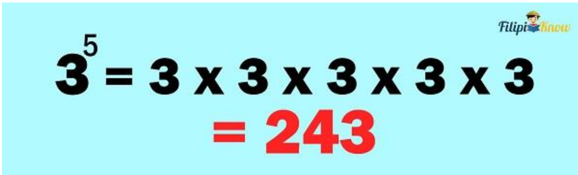
The exponent indicates how many times the base will be multiplied by itself. Thus, in 5^2, the exponent of 2 tells you that 5 is multiplied by itself two times.



Hence, $5^2 = 25$.

Example 1: *Compute* 3^5

Solution: The exponent of 5 tells us that 3 is multiplied by itself five times.



Therefore, $3^5 = 243$

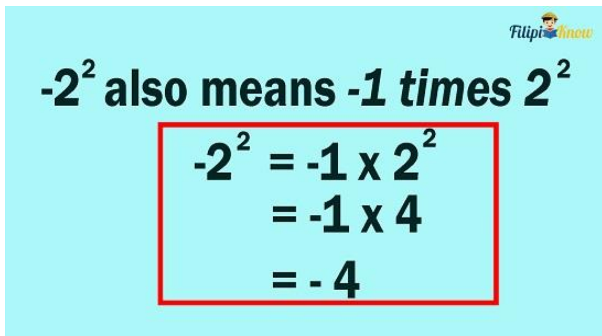
To generalize, given a^m where a and m are both real numbers, it means that a is multiplied to itself m times.

So far, we have tried to calculate numbers with exponents. However, all of our examples have positive bases. In our next section, let us discuss how to compute exponents of negative bases.

Computing Negative Numbers with Exponents.

Let's say we want to compute -2^2. This means that only 2 is being raised to the power of 2 and not -2. We can also interpret -2^2 as $- 1 \times 2^2$

Note that by applying the order of operations (PEMDAS), you need to perform exponents first before multiplication.



Hence, to compute -2^2, you first need to start computing for 2^2.

$2^2 = 4$

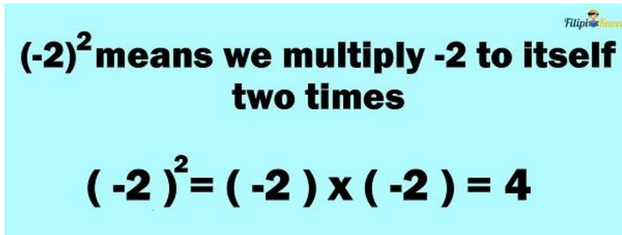
Afterward, multiply 4 by - 1:

$-2^2 = -4$

On the other hand, if we put a negative number inside a parenthesis, it means that we are raising the number together with the negative sign to the exponent.

For example, suppose we want to compute for $(-2)^2$. This means that - 2 is being raised to two.

Thus, to compute $(-2)^2$, we just multiply - 2 to itself two times:



Thus, $(-2)^2 = 4$.

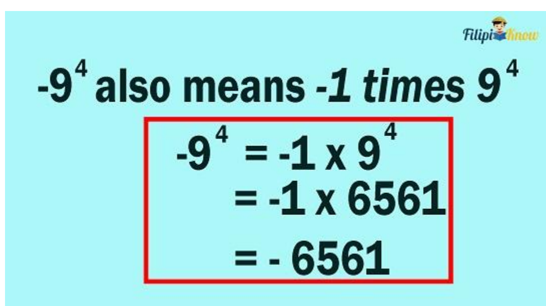
From the computations we have performed above, we can conclude that $-2^2 \neq (-2)^2$

Thus, a question that requires computing a negative number raised to an exponent may come in two forms. Here's how to solve each of them:

- **Case 1:** To compute $-a^b$ where a and b represent certain real numbers, we evaluate a^b first then multiply the result by - 1.
- **Case 2:** To compute $(-a)^b$ where a and b represent certain real numbers, we multiply $-a$ to itself b times.

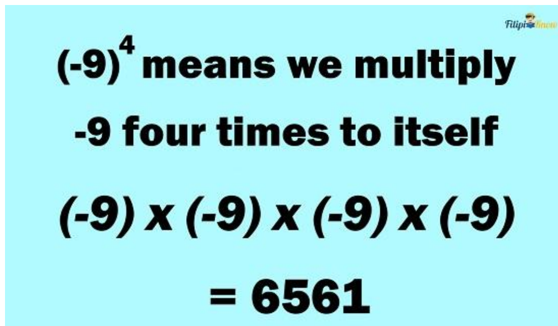
Example 1: *What is the value of -9^4?*

Solution: -9^4 is an example of Case 1. To compute for -9^4, we need to calculate 9^4 first. Afterward, multiply the result by -1:



Example 2: *What is the value of $(-9)^4$?*

Solution: Since - 9 is inside the parentheses, it indicates that - 9 is what is being raised to the power of 4. Thus, we need to multiply - 9 to itself four times.



Variables Raised to an Exponent.

When a variable has an exponent, it means that the variable is raised to a certain power. The exponent of the variable tells you how many times the variable is being multiplied by itself.

For instance, what does m^3 mean?

The exponent in m3 tells us that the variable m is raised to the power of 3. In other words, it tells us that m is being multiplied by itself 3 times.

$m^3 = m \cdot m \cdot m$

Note: *The solid dot as shown above is one of the alternative ways to express multiplication.*

Example 1: *Write k^5 in expanded form.*

Solution: The exponent in k^5 tells us that the variable k is being multiplied by itself 5 times. Thus, the expanded form of k^5 is:

$k^5 = k \cdot k \cdot k \cdot k \cdot k$

Example 2: *Express $u \cdot u \cdot u \cdot u \cdot u \cdot u$ in exponential form.*

Solution: Note that the variable *u* is used six times. Hence, we must use an exponent of 6. Thus:

$u \cdot u \cdot u \cdot u \cdot u \cdot u = u^6$

Variables with Coefficients.

Take a look at this algebraic expression: $5m^3$

Recall that *5* is the numerical coefficient of m^3 since it is a number multiplied by a variable.

Now, what does the exponent of 3 tell us in $5m^3$? What is the base of that exponent?

If you look closely at $5m^3$, the variable *m* is the only one raised to the power of 3, and 5 is not included. Hence, the base of the exponent 3 in $5m^3$ is *m* only and not *5m*.

Thus $5m^3$ means $5(m \cdot m \cdot m)$

Now take a look at this algebraic expression: $(5m)^3$

What does the exponent of 3 tell us in $(5m)^3$? What is the base of that exponent?

This time, the base is *5m*. It means that *5m* is being multiplied by itself three times.

Thus, $(5m)^3$ means $5m \cdot 5m \cdot 5m$

As a preview of what we have done above:

$5m^3 = 5(m \cdot m \cdot m)$

In this case, the numerical coefficient is **not included** in the base of the exponent.

$(5m)^3 = 5m \cdot 5m \cdot 5m$

In this case, the numerical coefficient is **included** in the base of the exponent.

When expanding a coefficient and variable raised to an exponent, check first what is the base of the exponent. Determine whether the coefficient is included in the base of the exponent or not.

Example 1: *Write $-3x^5$ in expanded form.*

Solution: The variable x is the only one raised to the power of 5. Thus, only the variable x is the base of exponent 5 in the given, and -3 is not included.

Thus, $-3x^5 = -3(x \cdot x \cdot x \cdot x \cdot x)$

Example 2: Write $(-3x)^5$ in expanded form.

Solution: The existence of the parentheses indicates that both the -3 and x in $-3x$ are raised to the power of 5. Thus, $-3x$ is the base of exponent 5.

In other words, $(-3x)^5 = -3x \cdot -3x \cdot -3x \cdot -3x \cdot -3x$

Laws of Exponents.

There are a lot of computations involving exponents that you will encounter as you study algebra. However, there are laws or certain rules that must be observed so you can perform these computations correctly. These laws are referred to as the *Laws of Exponents*.

Let us discuss these laws in this section one by one.

1. Product Rule.

Suppose we want to multiply x^2 by x^4. Note that x^2 and x^4 have the same base. *How can we multiply them?*

One possible method is to expand x^2 and x^4:

$x^2 = x \cdot x$

$x^4 = x \cdot x \cdot x \cdot x$

Multiplying the expanded values:

$x^2 \cdot x^4$

$(x \cdot x) \cdot (x \cdot x \cdot x \cdot x)$

Note that we can express the product of the expanded values into exponential form:

$(x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$

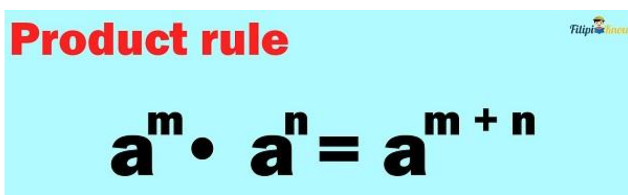
Therefore, $x^2 \cdot x^4 = x^6$

What have you noticed? Have you noticed a relationship between the exponents in $x^2 \cdot x^4 = x^6$?

Yes, the exponent in x^6 is just the sum of the exponents of x^2 and x^4.

You have now an idea of what the first law of exponent is all about. That is,

Product Rule: ***When multiplying exponential expressions that have the same base, copy the common base and add the exponents***.



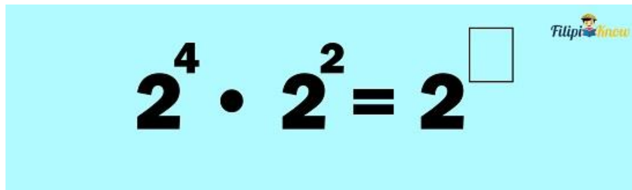
Thus, when multiplying expressions with the same base, you do not have to expand the given expressions to determine the answer. Just apply the product rule.

Keep in mind that **you cannot apply the product rule if the given bases are not the same**. For example, if you are going to multiply a^2 by p^3, you cannot apply the product rule since the given bases are not the same.

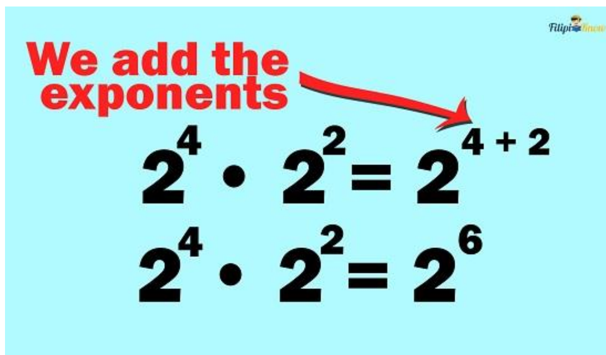
Example 1: *Compute for* $2^4 \cdot 2^2$

Solution: We have expressions with the same bases (i.e., 2) being multiplied together. Thus, we can apply the product rule.

Let us copy the common base first:



Then, add the exponents:

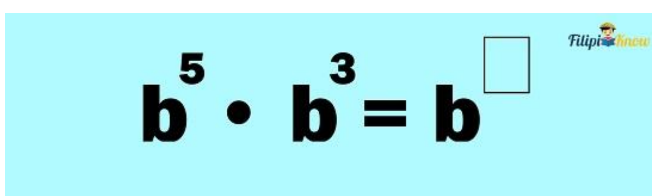


Thus, using the product rule: $\mathbf{2^4 \cdot 2^2 = 2^6}$

Example 2: *Multiply b^5 by b^3*

Solution: Since we have the same bases being multiplied together, we can apply the product rule:

Let us copy the common base first:



Then, add the exponents:

$$b^5 \cdot b^3 = b^{5+3}$$
$$b^5 \cdot b^3 = b^8$$

Therefore, using the product rule: $\boldsymbol{b^5 \cdot b^3 = b^8}$

Example 3: *Multiply* a^3b^2 *by* a^2b^4

Solution: We have two bases involved here, the variables *a* and *b.*

Thus, we need to apply the product rule, each for *a* and *b*:

Let us copy the common bases first:

$$a^3b^2 \cdot a^2b^4 = a^{\square}b^{\square}$$

Add the exponents for the common bases.

$$a^3b^2 \cdot a^2b^4 = a^{3+2}b^{2+4}$$
$$a^3b^2 \cdot a^2b^4 = a^5b^6$$

Hence, $a^3b^2 \cdot a^2b^4 = a^5b^6$

Example 4: *Multiply* $(x + 5)^6$ *by* $(x + 5)^3$

Solution: In this case, the common base is $x + 5$. Hence, we can apply the product rule.

Let us copy the common base first:

$$(X + 5)^6 \cdot (X + 5)^3 = (X + 5)^{\square}$$

Add the exponents:

$$(x + 5)^6 \cdot (x + 5)^3$$
$$= (x + 5)^{6+3}$$
$$= (x + 5)^9$$

Therefore, $(x + 5)^6 \cdot (x + 5)^3 = (x + 5)^9$

Example 5: *Compute for* $a(a^2)$

Solution: If two variables are written together with the other one enclosed in parentheses, it implies that the variables are being multiplied. Since we have a common base in the given (which is a), we can apply the product rule here.

Note that if a number or a variable has no exponent written above it, it implies that the exponent is 1.

Let us copy the common base first:

$$a\,(a^2) = a^{\square}$$

Add the exponents:

$$a\,(a^2) = a^{1+2}$$
$$a\,(a^2) = a^3$$

Therefore, $\boldsymbol{a(a^2) = a^3}$

2. Quotient Rule.

The Quotient Rule is the opposite of the Product Rule.

It states that **if you divide exponential expressions with the same base, you can just simply copy the common base then subtract the exponents.**

Quotient rule

$$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$$

Example 1: *Compute for* $x^7 \div x^3$

Solution: Since we are dividing exponential expressions with the same base, we can apply the quotient rule.

Let us copy the common base first:

$$x^7 \div x^3 = x^{\square}$$

Subtract the exponents:

$$x^7 \div x^3 = x^{7-3}$$
$$x^7 \div x^3 = x^4$$

Therefore, $\mathbf{x^7 \div x^3 = x^4}$

Example 2: *Simplify* x^9/x^4

Solution: x^9/x^4 also means $x^9 \div x^4$. Since we are dividing exponential expressions with the same base, we can apply the quotient rule.

Let us start by copying the common base:

$$\frac{x^9}{x^4} = x^{9-4} = x^{\square}$$

Finally, subtract the exponents:

$$\frac{x^9}{x^4} = x^{9-4} = x^5$$

Therefore, $x^9/x^4 = x^5$

Example 3: *Simplify* p^8q^2/p^6q

Solution: We have two bases involved: the variables *p* and *q*. Thus, we will use the quotient rule for the variables *p* and *q*.

Copying the common bases:

$$\frac{p^8q^2}{p^6q} = p^{\square}q^{\square}$$

Finally, subtract the exponents for each of the common bases.

$$\frac{p^8\ q^2}{p^6\ q} = p^{8-6}q^{2-1}$$

$$\frac{p^8\ q^2}{p^6\ q} = p^2q$$

Hence, $p^8q^2/p^6q = p^2q$

Example 4: *Divide 1 000 000 000 by 1 000 000*

Solution: Note that we can express 1 000 000 000 as 10^9. On the other hand, we can express 1 000 000 as 10^6. Therefore, we can answer the problem by dividing 10^9 by 10^6.

Since we have a common base (which is 10), we can apply the quotient rule:

Let us copy the common base first:

$$\frac{10^9}{10^6} = 10^{\boxed{}}$$

Then, subtract the exponents:

$$\frac{10^9}{10^6} = 10^{9-6}$$
$$= 10^3$$

Therefore, the answer is 10^3 or 1000.

Tip: *We can express a multiple of 10 into exponential form quickly by counting the number of zeros it has. For example, 1 000 000 000 has 9 zeros. Thus, if we express 1 000 000 000 in exponential form, we can determine the exponent to be used based on the number of zeros it has. Therefore 1 000 000 000 =* 10^9

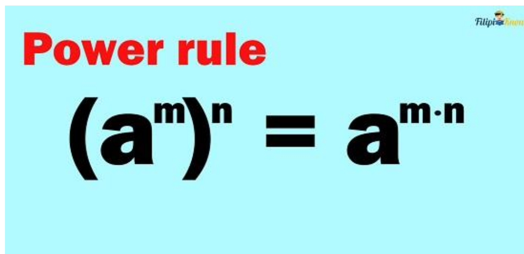
3. Power Rule.

Imagine a number raised to an exponent, then it's raised to another exponent. *What do you think will happen?*

Suppose we have b^2 and we want to raise it, say to the power of 3. This will give us $(b^2)^3$

Can we express $(b^2)^3$ using a single exponent only? The power rule states that we can!

Power rule states that if a number is raised to an exponent and then all raised to another exponent, you can combine the exponents into one by multiplying them.



Example 1: *Simplify $(k^4)^2$*

Solution: Notice that the entire k^4 is raised to 2. Applying the power rule, we can combine the exponents into one by multiplying them. Thus,

$(k^4)^2 = k^{4 \times 2} = k^8$

Therefore, $\mathbf{(k^4)^2 = k^8}$

Example 2: *What is the value of $(3^2)^3$?*

Solution: Applying the product rule:

$(3^2)^3 = 3^{2 \times 3} = 3^6$

Now, all we need to do next is expand 3^6:

$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 729$

Therefore, $\mathbf{(3^2)^3 = 729}$

Example 3: *Simplify $(a^5)^2$*

Solution: Applying the product rule:

$(a^5)^2 = a^{5 \times 2} = a^{10}$

Therefore, $(a^5)^2 = a^{10}$

Example 4: *Simplify* $(8y^5)^2$

Solution: Note that the base of the exponent 2 is $8y^5$. This means that we need to apply the power rule both for 8 and y^5:

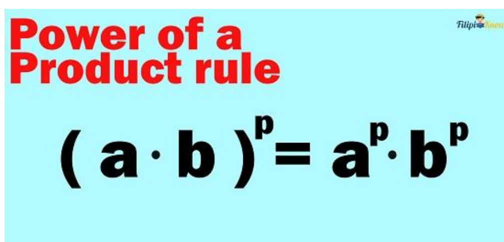
$(8y^5)^2 = 8^2y^{5 \times 2} = 8^2y^{10}$

Since $8^2 = 8 \cdot 8 = 64$, then $8^2y^{10} = 64y^{10}$

Therefore, $(8y^5)^2 = 64y^{10}$

4. Power of a Product Rule.

If an expression has more than one variable multiplied together and raised to a certain power, we can simplify that expression using the power of the product rule. This rule allows us to raise the variables involved in the multiplication process to the given exponent.



Example 1: *Simplify* $(xy)^2$

Solution: The given expression has two variables multiplied together (which is *xy)* and raised to the power of 2. This means that we can simplify it using the power of a product rule.

The power of the product rule allows us to “distribute” the exponent to each variable:

$(x\,y)^2 = (x^2)(y^2)$

Therefore, $\mathbf{(xy)^2 = x^2\,y^2}$

Example 2: *Simplify $(a^4b^3)^3$*

Solution: Let us apply the power of the product rule to simplify the given expression.

We start by "distributing" 3 to each of the variables:

$(a^4b^3)^3 = (a^4)^3\ (b^3)^3$

To further simplify the expression, we can apply the product rule to each variable:

$(a^4)^3\ (b^3)^3 = a^{4 \times 3}\ b^{3 \times 3} = a^{12}b^9$

Hence, $\mathbf{(a^4b^3)^3 = a^{12}b^9}$

Example 3: *Simplify $(4a^3b^2)^2$*

Solution: Let us apply the power of the product rule to simplify the given expression. Note that we should also raise 4 to the power of 2:

$(4a^3b^2)^2 = (4)^2\ (a^3)^2\ (b^2)^2$

Now, we apply the product rule to each variable:

$(4)^2(a^6)(b^4)$

Lastly, since $4^2 = 16$:

$16a^6b^4$

Therefore, $(4a^3b^2)^2 = 16a^6b^4$

5. Power of a Quotient Rule.

The power of the quotient rule is the opposite of the previous rule. It tells us that **if variables are divided together and raised to a certain power, we can simplify the expression using the power of the quotient rule.**

Power of a Quotient rule

$$\left(\frac{a}{b}\right)^{p} = \frac{a^{p}}{b^{p}}$$

Suppose we have the expression m/n and we want to raise it to the power of 3: $(m/n)^3$

Since we have two variables (*m* and *n*) divided together and raised to a certain power, we can apply the power of the quotient rule:

This rule allows us to raise the variables involved in the division process to the given exponent.

$(m/n)^3 = (m^3/n^3)$

Example 1: *Apply the power of the quotient rule to* $(x/y)^2$

Solution: Through the power of the quotient rule, we can distribute the exponent to the variables involved in the division process:

$$\left(\frac{x}{y}\right)^{2} = \frac{x^{2}}{y^{2}}$$

Therefore, using the power of the quotient rule, $(x/y)^2 = x^2/y^2$

Example 2: *Simplify* $(a^4/b^2)^2$

Solution: Since we have two variables (a^4 and b^2) divided together and raised to a certain power, we can apply the power of the quotient rule:

$$\left(\frac{a^4}{b^2}\right)^2 = \frac{(a^4)^2}{(b^2)^2}$$

Notice that we can simplify the expression further using the product rule:

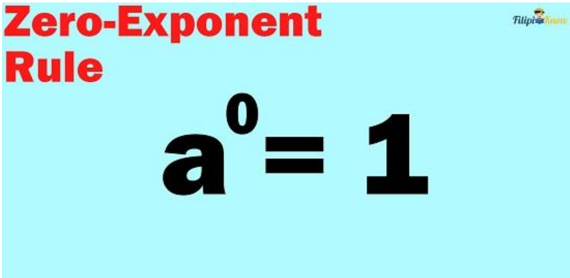
$$\frac{(a^4)^2}{(b^2)^2} = \frac{a^{4\cdot 2}}{b^{2\cdot 2}} = \frac{a^8}{b^4}$$

Therefore, $(a^4/b^2)^2 = a^8/b^4$

6. Zero-Exponent Rule.

What happens if you raise a number or a variable to the power of zero?

The zero-exponent rule tells us that the result will be 1.



The Zero-Exponent Rule states that any nonzero base raised to 0 is equal to 1.

Example 1: *Suppose that $m \neq 0$, what is the value of m^0?*

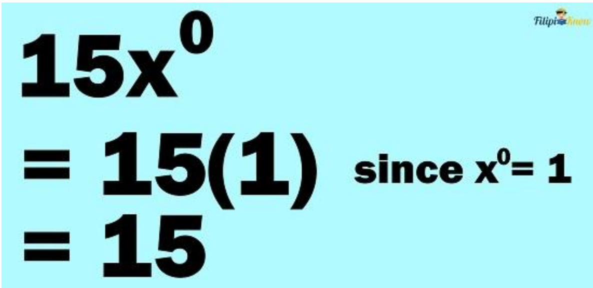
Solution: By the zero-exponent rule, $m^0 = 1$.

Example 2: *What is the value of 109^0?*

Solution: By the zero-exponent rule, $109^0 = 1$

Example 3: *Simplify $15x^0$*

Solution: We know that by the zero-exponent rule, $x^0 = 1$. Take note that x^0 is multiplied by 15 in $15x^0$. Since $x^0 = 1$:



Hence, $15x^0 = 15$

Example 4: *Simplify a^0b^2c*

Solution: By the zero-exponent rule, $a^0 = 1$. Since a^0 is multiplied to b^2c in the given expression a^0b^2c:

$$a^0b^2c$$
$$=(1)b^2c \quad \text{since } a^0 = 1$$
$$=b^2c$$

Hence, $a^0b^2c = b^2c$

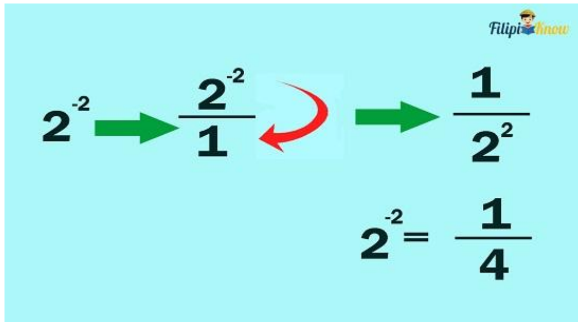
7. Negative Exponent Rule.

The negative exponent rule states that if a base is raised to a negative number, the base should be put to the denominator and the given negative exponent changed into a positive exponent.

Negative exponent rule

$$a^{-m} = \frac{1}{a^m}$$

Let's apply the negative exponent rule to 2^{-2}. Using the rule, we can put the base (which is 2) to the denominator. Note that the denominator of 2^{-2} is 1. Once we put the base into the denominator, we can change the negative exponent into a positive exponent.



Therefore, $2^{-2} = \frac{1}{4}$

Let us take a look at the given examples below to further understand this rule:

Example 1: *What is the value of 5^{-3}?*

Solution: Using the negative exponent rule, we can express 5^{-3} as $1/5^3$. Note that we can expand 5^3 as 5 x 5 x 5 and obtain 125. Therefore, $5^{-3} = 1/125$

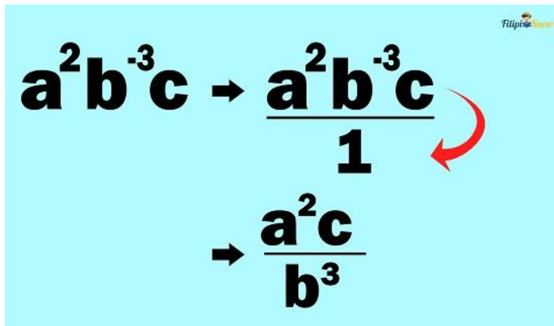
Example 2: *Express y^{-1} as an expression without a negative exponent.*

Solution: Using the negative exponent rule, we can express y^{-1} as $1/y$ which has no negative exponent involved.

Thus, the answer is $1/y$.

Example 3: *Express $a^2b^{-3}c$ without a negative exponent.*

Solution: We can apply the negative exponent rule to express $a^2b^{-3}c$ without a negative exponent. However, since b^{-3} is the only base raised to a negative exponent, then we can only apply the negative exponent rule to b^{-3} and it is the only base that we are going to put in the denominator.



Therefore, $a^2b^{-3}c$ can be written without a negative exponent as a^2c/b^3

Laws of Exponents Summary.

We are now done discussing the mathematical rules that govern the exponential expressions. As a recap, here's a table that summarizes the laws of exponents.

Product Rule	$a^m \cdot a^n = a^{m+n}$
Quotient Rule	$a^m/a^n = a^{m-n}$
Power Rule	$(a^m)^n = a^{mn}$
Power of a Product Rule	$(a\,b)^p = a^p\,b^p$
Power of a Quotient Rule	$(a/b)^m = a^m/b^m$, where $b \neq 0$
Zero Exponent Rule	$a^0 = 1$
Negative Exponent Rule	$a^{-m} = 1/a^m$

Simplifying Exponential Expressions Using the Laws of Exponents.

An exponential expression is simplified if there are fewer terms and exponents involved. Furthermore, a simplified exponential expression has positive exponents.

As we simplify various exponential expressions, we have to apply different laws of exponents that we have discussed above.

Example 1: *Simplify* $5p^0q^{-2}$

Solution: We can simplify the given expression by making all of its exponents positive.

Let us start by applying the zero-exponent rule:

$\boldsymbol{5p^0q^{-2}}$

$\boldsymbol{5(1)q^{-2}}$ (since $p^0 = 1$)

$\boldsymbol{5q^{-2}}$

We can then remove the negative exponent using the negative exponent Rule:

$5q^{-2}$

$5/q^2$

Thus, the answer is $5/q^2$

Example 2: *Simplify* $(a^4/a^2)^2$

Solution 1: Note that we can distribute the exponent 2 which is outside the parentheses to the bases that are inside the parentheses using the power of the quotient rule:

$(a^4/a^2)^2 = a^{4\times 2}/a^{2\times 2} = a^8/a^4$

Since we are dividing the same bases, we can apply the quotient rule:

$a^8/a^4 = a^{8-4} = a^4$

Therefore, $(a^4/a^2)^2 = a^4$

You can also simplify the given expression using the alternative solution below.

Solution 2: This time, let us start applying the quotient rule since we are dividing the same bases:

$(a^4/a^2)^2 = (a^{4-2})^2 = (a^2)^2$

Notice that we can apply now the power rule since $(a^2)^2$ is an expression raised to an exponent then raised to another exponent.

$(a^2)^2 = a^{2 \times 2} = a^4$

Example 3: *Simplify* $[(x + y)^2(x + y)^3]^{-1}$

Solution: We can start by making the negative exponent positive. To do this, put the base into the denominator (negative exponent rule). The base in the given expression is the entire $(x + y)^2(x + y)^3$

$$[(x+y)^2(x+y)^3]^{-1} \rightarrow \frac{1}{(x+y)^2(x+y)^3}$$

We can simplify the expression further using the product rule since we are multiplying the same bases:

$$\frac{1}{(x+y)^2(x+y)^3} = \frac{1}{(x+y)^{2+3}}$$

$$= \frac{1}{(x+y)^5}$$

Therefore, the answer is $1/(x+y)^5$