So far, we have explored the world of one and two-dimensional geometric figures such as lines, angles, and plane figures. This time, we are going to the realm of three-dimensional geometric figures called solid figures.

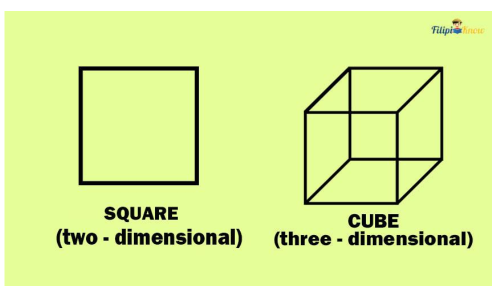
Objects around you are mostly solid figures. The ice cube you use for your juice, the basketball you dribble, the traffic cone you see on the road, and a lot more! It will take you forever to list all of them.

This chapter will delve into the world of solid geometric figures such as cubes, cones, pyramids, rectangular prisms, and cylinders. We are also going to learn how to compute for their respective volumes and use this technique to solve some real-life word problems.

What Are Solid Figures?

Solid figures are geometric figures that have three dimensions - width, depth, and height. We are living in a three-dimensional world where almost every object around us is three-dimensional.

Take a look at the given image below. The geometric figure on the left side is a two-dimensional square (i.e., a plane figure). Meanwhile, the one you see on the right side is a three-dimensional cube. Can you see how these figures are different from each other?



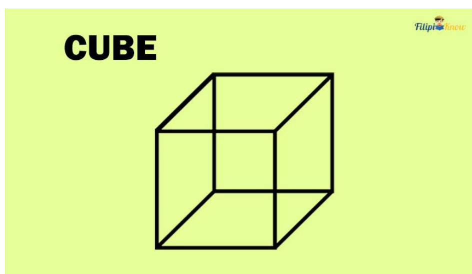
A cube is an example of a solid figure since it is three-dimensional.

In this module, we shift our focus to the branch of geometry that deals with solid figures, the solid geometry.

What Are the Types of Solid Figures?

Let us discuss the most common types of solid figures we can observe in the real world.

1. Cube



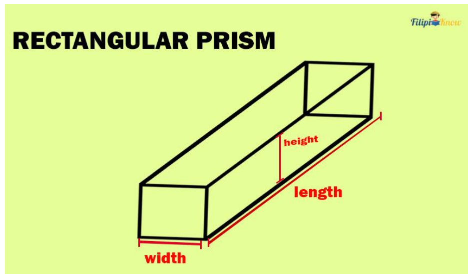
A cube is a solid geometric figure with six squares serving as its "faces."

The image above shows an example of a cube. Notice that it is made by squares joined together to form this solid figure. These squares are the faces of the cubes. One of the best examples of a cube is the popular puzzle "Rubik's cube," which most of us are familiar with.

Other examples of a cube are ice cubes, sugar cubes, and the six-sided dice used in board games.

Since cubes have square faces, the sides of a cube are all congruent or have the same measurement.

2. Rectangular Prism



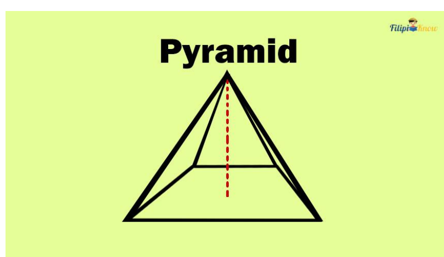
A rectangular prism has six faces, just like a cube. But instead of squares, its faces are all rectangles.

A rectangular prism has **length, width, and height.** The length of the rectangular prism refers to how long it is, the width refers to how wide or thick it is, and the height refers to how tall it is.

Rectangular prisms are also called cuboids or rectangular solids.

Examples of a rectangular prism are aquariums (or fish tanks), shoeboxes, cabinets, books, cargo containers, and a lot more.

3. Pyramid



A pyramid is a solid figure with triangular sides that meet on a certain point called the apex and a base that is a polygon.

Notice that the square pyramid above has three triangular sides that meet at a certain point (which is the apex). Its base is a square, which is why we call it a square pyramid.

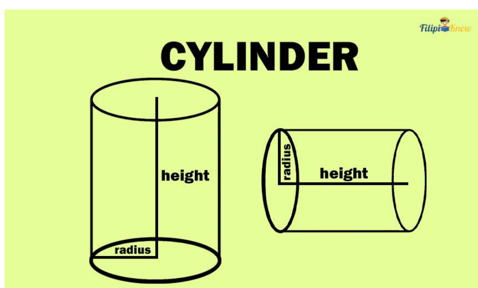
The vertical line that connects the base and the apex (topmost point) of the pyramid is the height of the pyramid. In the figure above, the red line represents the height.

We have a triangular pyramid if we create such a pyramid with a triangular base. We have a hexagonal pyramid if you use a hexagon as the base. In other words, you can use any polygon as a base to create a pyramid.

The real-life example of a pyramid that we are all familiar with is the Egyptian Pyramids. These structures were built thousands of years ago and are considered one of the most beautiful creations of ancient human civilizations. It is considered one of the world's seven ancient wonders.

Other real-life examples of a pyramid include tents and some perfume bottles.

4. Cylinder

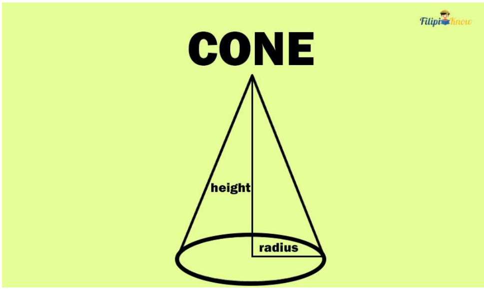


A cylinder is a solid figure with two ends that are congruent circles and parallel. A curved surface connects these circles.

A cylinder has a radius located in its circular ends and a height which is the vertical distance from the center of one circular end to another. The height of the cylinder tells you how long (if the cylinder is horizontally placed) or how high (if the cylinder is vertically placed) it is.

One of the best visual examples of a cylinder is a tin can. A tin can has two ends, both circles connected by a curved surface. Aside from tin cans, batteries, cylindrical tanks, drinking glasses, toilet paper rolls, and beakers are some of the real-life examples of cylinders.

5. Cone

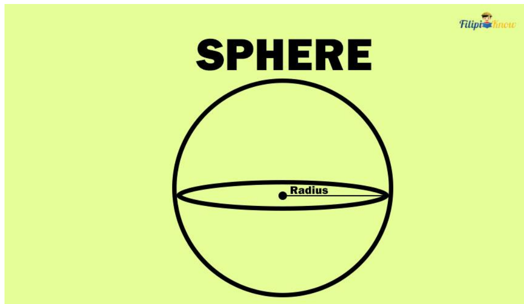


A cone has a circular (or elliptical) base and a vertex point that is connected by a curved surface.

As you can see in the image above, a cone has a circular base that is connected to the topmost point (called apex) through a curved surface. A cone has a radius located in its circular base and a height which is the vertical distance from the apex to the center of the circle.

It is not that difficult to imagine what a cone looks like in real life since it best resembles the ice cream cone we use for our *sorbetes*. You also see them always on the road in the form of traffic cones which help in traffic management. And who can't forget those conical birthday hats we used to wear during birthday parties?

6. Sphere



The sphere is the solid figure counterpart of a circle. It is the set of all points that are equidistant or have the same distance from a center in a three-dimensional space. In other words, you can think of spheres as solid round figures.

Just like a circle, the sphere has a radius. The sphere's **radius** is the distance from the sphere's center to the point on the sphere. Furthermore, if we form a line segment from one point on the sphere to another and let it pass through the center, we form the **diameter** of the sphere. **The length of the diameter of a sphere is twice as long as the length of its radius**, just like in a circle.

Real-life examples of a sphere include globes that we use to represent the planet Earth, the balls that we use in outdoor sports activities such as basketball and volleyball, and those marbles (or *jolens*) that we loved collecting as a child.

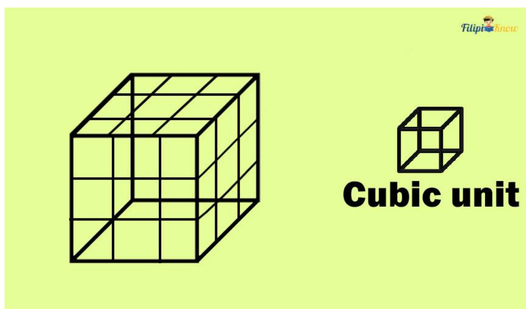
There are more solid figures, but we will focus only on these six common solid figures.

Volume of a Solid Figure

The space that a solid figure occupies is its volume.

Think of an aquarium or a fish tank. We know that a fish tank is an example of a solid figure (specifically, a rectangular prism). The maximum amount of water you can put in that fish tank is equivalent to the volume of that tank.

As shown in the image below, you can think of the volume as the number of cubic units that you can put inside a solid figure.



However, in real life, you cannot see cubic units inside a solid figure. Thus, we must use reliable and accurate methods to determine the volume of solid figures. For this reason, mathematicians derived various formulas on how to find the volume of these solids.

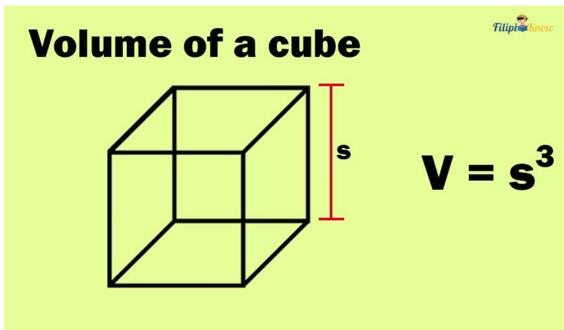
In this section, let us talk about these methods and formulas.

How To Compute the Volume of Solid Figures

Before we proceed to calculate the volumes of various solid figures, make sure that you have mastered the operations on fractions and decimals because you'll be applying them in the succeeding calculations. If you feel you need to brush up on your manual calculation skills, we advise you to do it first.

Once you are ready, read the succeeding sections on how to find the volume of solid figures.

1. Volume of a Cube



The volume of a cube is just the measurement of the cube's side multiplied by itself three times. In other words, the volume of the cube can be obtained by raising the length of the side to the third power or by "cubing" it.

In symbols:

$$V_{cube} = s^3$$

Quite simple, right?

Sample Problem 1: A cube has a side with a length of 3 cm. Determine the volume of the cube.

Solution: As we have stated earlier, we can get the volume of this cube just by multiplying the length of the side three times to itself or raising it to the third power.

The given side of the cube is 3 cm long. Therefore, we have s = 3.

Using the formula:

$$V_{cube} = s^3 = (3)^3 = 3 \times 3 \times 3 = 27$$

Hence, the volume of the cube is 27 cm^3.

Take note that when we are writing the volume of a solid figure, we express the given units in "cubic units." This means that if the given measurement is in centimeters (cm), the volume must be written in cubic centimeters (cm^3). Likewise, if the given measurement is in feet (ft), the volume must be cubic feet (ft^3).

Sample Problem 2: The packaging used by Lemon Inc. for its lemon-flavored juice drink is cubic in shape. Suppose that the side of this cubic packaging material is 15 cm long. How many cubic centimeters of juice drink can be put in the packaging?

Solution: To determine the number of cubic centimeters of juice drink that can be put in the packaging, we must determine its volume first. Since the packaging is cubic in shape, we can use the formula for the volume of a cube.

$$V = s^3$$

The side of the packaging is 15 cm long. Thus, we have s = 15.

Let us substitute s = 15 into the formula:

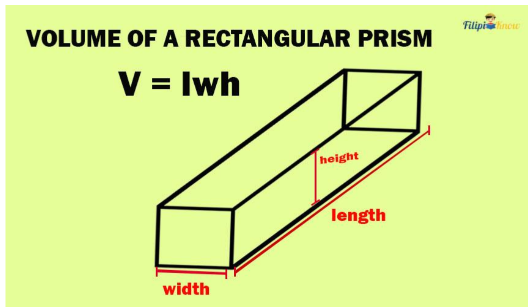
$$V = (15)^3$$

$$V = 15 \times 15 \times 15$$

$$V = 3375$$

Thus, the volume of the cubic packaging is 3375 cm^3. It means that about 3375 cm^3 of the juice drink can be put in the packaging.

2. Volume of a Rectangular Prism



Recall that a rectangular prism has length, width, and height.

Computing the volume of a rectangular prism is quite simple. All you have to do is multiply the length, width, and height of the rectangular prism.

Mathematically,

$$V_{rectangular\ prism} = lwh$$

where *l* stands for the length, *w* stands for the width, and *h* stands for the height of the rectangular prism.

Here are some examples of how to find the volume of a rectangular prism:

Sample Problem 1: What is the volume of a shoebox that is 4 inches long, 3 inches wide, and 3 inches high?

Solution: A shoebox is an example of a rectangular prism. Since we have the shoebox's length, width, and height, we can compute its volume.

The volume of a rectangular prism is just $V = lwh$ or the product of its length, width, and height.

We have l = 4, w = 3, and h = 3. Using the formula:

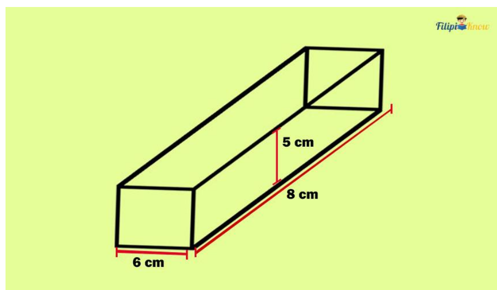
$$V = lwh$$

$$V = (4)(3)(3)$$

$$V = 36$$

Thus, the volume of the shoebox is 36 cubic inches (in^3).

Sample Problem 2: Compute the volume of the rectangular prism below.



Solution: The given figure above tells us that the length of the rectangular prism is 8 cm, its width is 6 cm, and its height is 5 cm. Recall that the volume of a rectangular prism is just the product of the measurements of its length, width, and height:

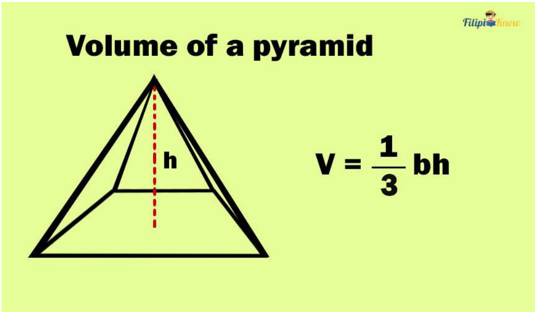
$$V = lwh$$

$$V = (8)(6)(5)$$

$$V = 240$$

Thus, the volume of the rectangular prism is 240 cm^3.

3. Volume of a Pyramid



The volume of a pyramid is equivalent to ⅓ of the product of the area of its base and its height.

$$V = \frac{1}{3} bh$$

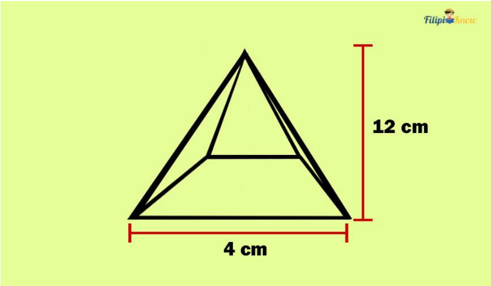
where *b* stands for the area of the polygonal base of the pyramid and *h* stands for its height.

Most of the time, the height of the pyramid is given. However, the value of the area of the pyramid's base depends on the base type.

If we have a square pyramid (a pyramid with a square base), then the area of the pyramid's base is equivalent to the area of the square used as the base. Meanwhile, if we have a triangular pyramid (a pyramid with a triangular base), then the area of the base of the pyramid is the area of that triangular base.

Thus, **the area of the pyramid's base is equal to the area of the polygon it uses as the base.** To further understand this concept, let us solve some examples.

Sample Problem 1: Determine the volume of the pyramid below:



Solution: We have a square pyramid above since the base is a square. Furthermore, the height is already given, which is 12 cm. Furthermore, the side of the square base is given, which is 4 cm. Let us determine the area of this square base so that we can find the volume of this pyramid.

The formula for the area of a square is $A = s^2$. The side of the pyramid's square base is 4 cm. Thus, we have s = 4.

Therefore, the area of the square is $A = (4)^2 = 16$. Thus, we will use b = 16 as the area of the base in our formula.

We now have base = 16 (as what we have computed above) and height = 12 cm (this is given). Let us now do the math and compute for the volume of the pyramid:

$$V = \frac{1}{3} bh$$

$$V = \frac{1}{3} (16)(12)$$

$$V = 64$$

Thus, the area of the square pyramid above is 64 cm^3.

Sample Problem 2: A pyramid has a rectangular base. The base has a length of 5 inches and a width of 3 inches. Determine the volume of the pyramid if it is 6 inches tall.

Solution: We already have the height of the pyramid, which is 6 inches. However, we still need to determine the area of the base for us to compute the volume of the pyramid.

How do we find the area of the base of this pyramid?

We have a pyramid that has a rectangular base. Therefore, we need to find the area of that rectangular base.

To clarify, we need to look for the area of the rectangular base so that we can use it in the formula for the volume of the pyramid.

The rectangular base has a length of 5 inches and a width of 3 inches. The area of a rectangle is computed as $A = lw$ or the product of the length and the width. Therefore, the area of the rectangular base is:

$$A = lw$$

$$A = 5(3) = 15\ cm^2$$

So, the area of the rectangular base is 15 cm^2. We now have a value of *b* for the formula.

Let us now compute the area of the pyramid using h = 6 (this is given) and b = 15 (as computed above).

$$V = ⅓\ bh$$

$$V = ⅓\ (15)(6)$$

$$V = 30$$

Therefore, the volume of the rectangular pyramid is 30 cm^3.

To make things easier, I will share with you some “specific” formulas that you can use instead to find the volumes of a square pyramid, rectangular pyramid, and triangular pyramid:

For square pyramid:

$$V_{square\ pyramid} = \frac{1}{3} s^2 h$$

where *s* is the side of the pyramid's square base, and *h* is the height.

For rectangular pyramid:

$$V_{rectangular\ pyramid} = \frac{1}{3} lwh$$

where *l* and *w* are the length and width of the rectangular base and *h* is the height of the pyramid.

For triangular pyramid:

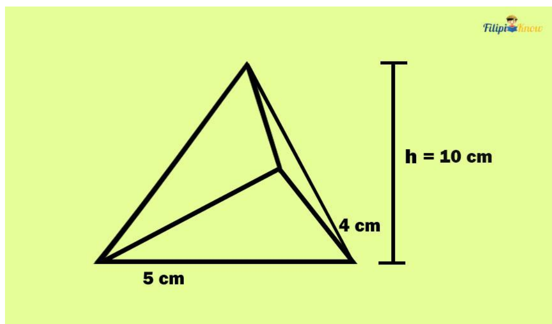
$$V_{triangular\ pyramid} = \frac{1}{6} (b_t h_t) h_p$$

where b_t and h_t are the base and the height of the triangular base, respectively, and h_p is the height of the pyramid.

I know that it is taxing to memorize these formulas, so I advise you to stick to the method we discussed earlier, which is to find the area of the polygonal base first of the pyramid and then compute its volume. However, if you opt to memorize these formulas, there's no harm in doing it since it takes less time to use these formulas than the long method.

Anyway, let us try to solve the volume of a triangular pyramid using both of these methods:

Sample Problem 3: A pyramid has a base in the shape of a triangle. The triangular base has a height of 5 cm and a base of 4 cm. Meanwhile, the pyramid is 10 cm high. Calculate the volume of the pyramid.



Method 1: Longer Method

Let us compute the area of the triangular base first.

The height of the triangular base is 5 cm while its base is 4 cm. Using the formula for the area of a triangle:

$$A = \frac{1}{2} bh$$

$$A = \frac{1}{2} (4)(5)$$

$$A = \frac{1}{2} (20)$$

$$A = 10$$

Thus, the area of the triangular base is 10 cm^2.

Now, we have base = 10 and height = 10.

Let us determine the volume of the pyramid using the "general" formula for the volume of a pyramid:

$$V = \frac{1}{3} bh$$

$$V = \frac{1}{3} (10)(10)$$

$$V = 100/3$$

$$V = 33.33$$

Thus, the volume of the triangular pyramid in the problem is approximately 33.33 cm^3.

Method 2: Using the "specific formula" for the triangular pyramid

$$V_{\text{triangular pyramid}} = \frac{1}{6} (b_t h_t) h_p$$

b_t stands for the base of the triangular base. So, we have b_t = 4 cm. Meanwhile, h_t represents the height of the triangular base. So, we have h_t = 5 cm. Lastly, h_p represents the height of the pyramid, so we have h_p = 10

So, to summarize, we have b_t = 4, h_t = 5, and h_p = 10

Using the formula above:

$$V_{\text{triangular pyramid}} = \frac{1}{6} (b_t h_t) h_p$$

$$V_{\text{triangular pyramid}} = \frac{1}{6} (4)(5)(10)$$

$$V_{\text{triangular pyramid}} = \frac{1}{6} (20)(10)$$

$$V_{\text{triangular pyramid}} = \frac{1}{6} (200)$$

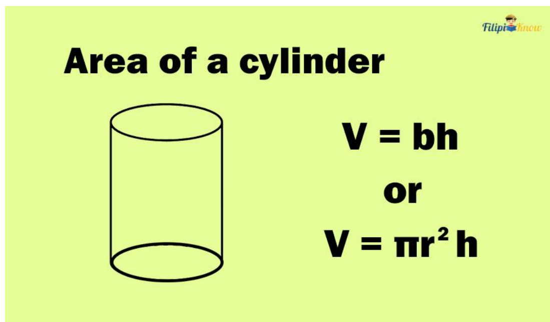
$$V_{\text{triangular pyramid}} = 200/6$$

$$V_{\text{triangular pyramid}} = 33.33$$

Thus, the volume of the triangular pyramid is 33.33 cm^3.

Notice that whether you use the longer method or the “specific” formula, you will still arrive at exactly the same answer. So, it is up to you which method you prefer is more convenient and practical.

4. Volume of a Cylinder



The volume of a cylinder is the product of the area of its circular base and its height.

In symbols,

$$V_{cylinder} = bh$$

where *b* is the area of the circular base and *h* is the height of the cylinder.

Take note that since a cylinder has a circular base, then the area of the circular base (b) is $b = \pi r^2$, where r is the radius of the circular base.

We can make the given formula above more specific as:

$$V_{cylinder} = \pi r^2 h$$

where r is the radius of the circular base.

You might ask this yourself after looking at the given formulas: *Which one should I use?*

If the circular base area is already given in the problem, then use V = bh. Meanwhile, if the area of the circular base is unknown and only the radius of the base is given, then use $V = \pi r^2 h$.

Sample Problem 1: The radius of the circular base of a cylindrical tank is 3 meters long. The cylindrical tank is 6 meters high. Determine the total amount of water that can be put in the cylindrical tank (Use π = 3.14).

Solution: The total amount of water that you can put inside the cylindrical tank depends on the volume of that cylindrical tank. So, to answer the given problem, we must compute the volume of the cylindrical tank.

The height of the cylindrical tank is 6 meters. Meanwhile, the radius of its circular base is 3 meters long. Since we have the radius of the circular base rather than its area, then we must use the formula $V = \pi r^2 h$.

Let us substitute r = 3 and h = 6 in the formula. Take note that the estimate of π that we are going to use is 3.14:

$$V = \pi r^2 h$$

$$V = (3.14)(3)^2(6)$$

$$V = (3.14)(9)(6)$$

$$V = (3.14)(54)$$

$$V = 169.56$$

Thus, the volume of the cylindrical tank is 169.56 m^3. This means that the total amount of water that can be poured inside is about 169.56 cubic meters.

Sample Problem 2: The diameter of the circular base of a can of milk is 10 cm long. If the can is 14 cm tall, determine the amount of milk that can be put in the can (Use π = 3.14).

Solution: The total amount of milk that can be put inside the can depends on its volume.

The circular base of the can of milk has a diameter of 10 cm long. Take note that the diameter is twice the measurement of the radius. Thus, if the diameter of the circular base is 10 cm, then the radius of the circular base must be 10/2 = 5 cm.

So, we have 5 cm as the radius of the circular base (r = 5).

Meanwhile, the problem states that the can is 14 cm tall. It means that we have 14 cm as the height of this cylindrical can (h = 14).

Since we already have the radius and height of the cylinder, let us use the formula $V = \pi r^2 h$.

$$V = \pi r^2 h$$

$$V = (3.14)(5)^2(14)$$

$$V = (3.14)(25)(14)$$

$$V = 1099$$

Thus, the volume of the can of milk is 1099 cm^3. It means that 1099 cubic centimeters of milk can be put inside the milk can.

Sample Problem 3: A clock's battery is cylindrical in shape. It has a base with an area of 2.5 cm^2 and a height of 5 cm. Determine the volume of the clock's battery.

Solution: The problem provided us with the area of the circular base of the cylinder and its height. Therefore, we can use the formula V = bh.

The given area of the base is 2.5 cm^2. So, we have b = 2.5.

Meanwhile, the height of the battery is 5 cm. So, we have h = 5.

Let us now substitute these values for the formula.

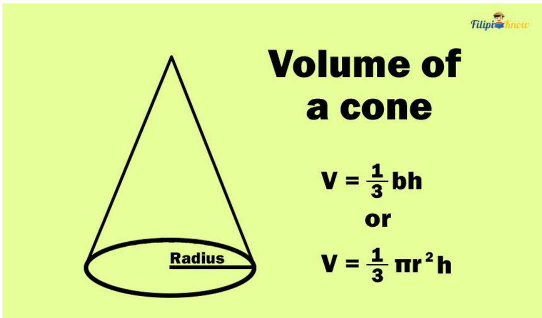
$$V = bh$$

$$V = (2.5)(5)$$

$$V = 12.5$$

Thus, the volume of the clock's battery is 12.5 cm^3.

5. Volume of a Cone



The volume of a cone can be calculated as ⅓ of the product of the area of its circular base and its height.

In symbols,

$$\mathbf{V_{cone} = \tfrac{1}{3}\, bh}$$

where *b* is the area of the cone's circular base and *h* is its height.

Since a cone has a circular base, then the area of the circular base (b) is $b = \pi r^2$ where *r* is the radius of the circular base of the cone.

Hence, we can write the formula above more precisely as:

$$\mathbf{V_{cone} = \tfrac{1}{3}\, \pi r^2 h}$$

where *r* is the radius of the circular base of the cone.

As you may have noticed, the formula for the volume of a cone is similar to the formula for the volume of a cylinder. The two solid figures almost have the same formula, except that the cone has an additional ⅓ in its formula. **In fact, if a cone and a cylinder have the same radius and height, then the volume of the cone is equal to ⅓ of the volume of the cylinder.**

Going back to how the volume of a cone is computed, if the area of the circular base and the height of the cone is already given in the problem, just use the formula V = ⅓ bh. On the other hand, if only the radius of the circular base is given and its height, then use $V = \frac{1}{3} \pi r^2 h$.

Let us try to solve some examples.

Sample Problem 1: A cone has a base with an area of 120 mm^2. If the cone is 50 mm high, determine the volume of the cone.

Solution: In this given problem, the height and the area of the base of the cone are given. Thus, we can use the formula V = ⅓ bh to determine its volume.

So, we have b = 120 and h = 50. Let us calculate the volume of the cone.

$$V = \frac{1}{3} bh$$

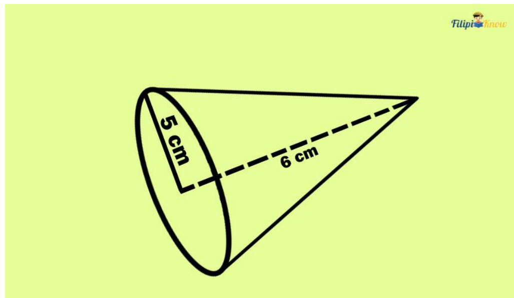
$$V = \frac{1}{3} (120)(50)$$

$$V = \frac{1}{3}(6000)$$

$$V = 2000$$

Hence, the volume of the cone is 2000 mm^3.

Sample Problem 2: Determine the volume of the cone below (Use as it is).



Solution: Based on the given illustration above, the cone has a defined height of 6 centimeters and a radius of 5 centimeters. Since the cone's height and radius are already given, it is easier to use the formula $V = \pi r^2 h$.

We have h = 6 and r = 5. Take note that the problem requires us to use π as it is, so we don't have to use an approximate value of it and just use the Greek letter.

$$V = \pi r^2 h$$

$$V = \pi(5)^2(6)$$

$$V = \pi(25)(6)$$

$$V = 150\pi$$

Therefore, the volume of the cone is 150 cm^2.

Sample Problem 3: Rose has a cylindrical pencil holder and a cone-shaped mini-lamp which have the same radius (of the circular base) and height. The cylindrical pencil holder has a height of 25 cm and a radius of 10 cm. Determine the volume of Rose's cone-shaped mini-lamp using the volume of Rose's cylindrical pencil holder.

Solution: Recall that we have stated earlier that the volume of a cylinder with the same height and radius as a cone is equivalent to ⅓ of the volume of the cone. Thus, to answer the given problem, we need to find first the volume of the cylindrical pencil holder and then take the ⅓ of it to obtain the volume of Rose's cone-shaped mini-lamp.

Let us derive first the volume of Rose's cylindrical pencil holder. The problem states that the pencil holder has a height of 25 cm and the radius of its circular base is 10 cm. Since we have the height and the radius of the circular base, it is more convenient to use the formula $V = \pi r^2 h$ for the volume of the cylinder.

We have r = 10 and h = 25. Take note that there's no estimate of π that we are required to use, so we just use it as it is. Substituting these values to the formula:

$$V_{cylinder} = \pi r^2 h$$

$$V_{cylinder} = \pi(10)^2(25)$$

$$V_{cylinder} = \pi(100)(25)$$

$$V_{cylinder} = 2500\pi$$

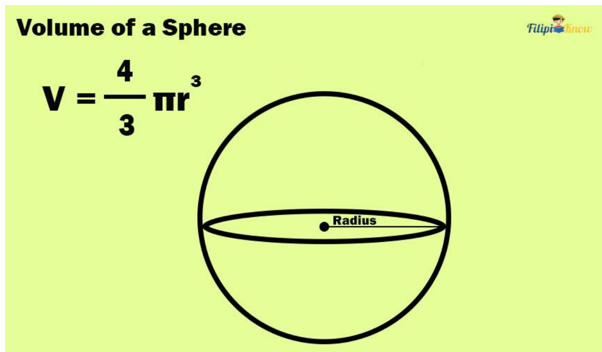
Thus, the volume of the cylindrical pencil holder is $2500\ \pi cm^3$. Now, we just need to take the ⅓ of the volume of the cylindrical pencil holder to obtain the volume of the cone-shaped mini-lamp. Thus:

$$V_{cone} = \tfrac{1}{3}(2500\pi)$$

$$V_{cone} = 833.33\pi$$

Thus, the volume of the cone-shaped mini lamp is $833.33\ cm^3$

6. Volume of a Sphere



The volume of a sphere can be obtained using the formula below:

$$\mathbf{V_{sphere} = 4/3\pi r^3}$$

where *r* is the radius of the sphere.

The formula above is actually derived by the Greek mathematician Archimedes who also provided one of the earliest estimates of π.

The volume of a sphere can be obtained by multiplying the cube of the sphere's radius by π, then multiplying the result by 4, and then dividing the result by 3.

Sample Problem 1: A ball has a radius of 2 meters. Determine the maximum amount of air that the ball can hold.

Solution: The amount of air that the ball can hold depends on its capacity or volume.

The radius of the ball is 2 meters. Thus, we have r = 2.

Take note that the problem did not provide us any approximate value of π that we are required to use, so we just use it as it is.

Using the formula for the volume of a sphere: V = $\mathbf{4/3\pi r^3}$

V = $\mathbf{\frac{4}{3}}\pi\mathbf{r^3}$

V = $\mathbf{\frac{4}{3}}\pi(2)^3$

V = $\mathbf{\frac{4}{3}}\pi(8)$

$$V = \frac{4\pi(8)}{3} = \frac{32\pi}{3}$$

Thus, the volume of the ball is $\frac{32\pi}{3}$ m^3. This implies that the amount of air it can hold is about $\frac{32\pi}{3}$ cubic meters (m^3).

Sample Problem 2: A newly bought globe has a diameter of about 20 inches. Determine the volume of that globe (Use = 3.14).

Solution: The given problem is the diameter of the globe which is 20 inches. However, the sphere's volume formula uses the radius and not the diameter. Thus, we have to determine the radius first.

Recall that the diameter of a round figure such as a sphere or a circle is twice as long as the radius. Thus, if the diameter of the globe is 20 inches, it means that its radius must be 20/2 = 10 inches.

Hence, we have r = 10 inches.

Let us now use the formula for the volume of a sphere.

$$V = \mathbf{\frac{4}{3}}\pi\mathbf{r^3}$$

$$V = \mathbf{\frac{4}{3}}(3.14)(10^3)$$

$$V = \mathbf{\frac{4}{3}}(3.14)(1000)$$

$$V = \frac{4}{3}(3140)$$

$$V = 4186.666\ldots$$

Thus, the volume of the sphere is 4186.67 cm^3.