Imagine yourself as a mathematician during the 16th century. This is an era without calculators, computers, and spreadsheets so dealing with calculations of numbers with multiple digits is tedious and time-consuming.

Things changed when a Scottish mathematician named John Napier came up with a revolutionary idea to ease the way calculations are made. He is one of the mathematicians who developed **logarithms**, a convenient tool that minimizes time spent on complex calculations.

Let us refresh our memory with the definition and properties of logarithms in this reviewer.

Logarithms

Logarithms are an alternative way of expressing quantities that involve exponents.

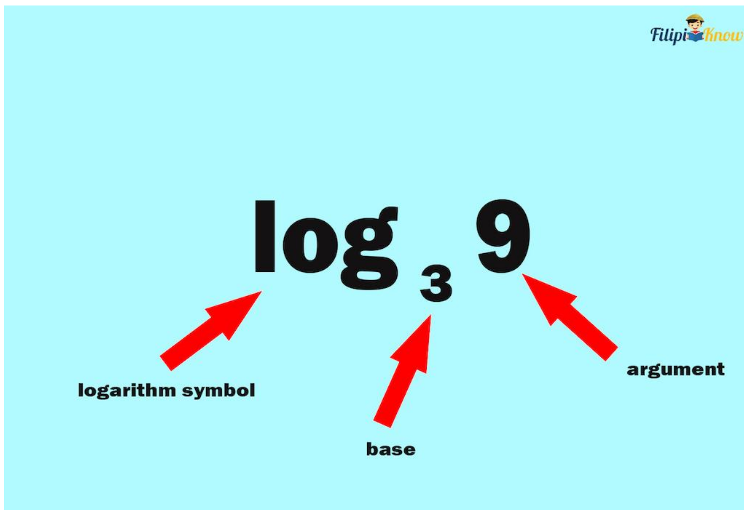
For instance, the mathematical sentence $3^2 = 9$ can be written in its equivalent logarithmic form, $\log_3 9 = 2$.

Recall that the exponent tells us the number of ways the base is used as a factor in a multiplication sentence For instance, $3^2 = 9$ means that we must use 3 as a factor twice in a multiplication sentence to get 9 (i.e., 3 x 3 = 9).

Meanwhile, logarithms tell us how many times a number must be multiplied by itself to obtain another number. For instance, $\log_3 9 = 2$ (read as *"log of 9 to the base 3 is equal to 2"*) tells us that if we use an exponent to transform 3 into 9, then we must use 2.

You will understand further how logarithms work once we discuss their components in the next section.

Components of a Logarithm



A logarithm has the following major components:

1. The **"log"** function, which states that we are dealing with a logarithm
2. The **base** or the small number below the "log" symbol
3. The **argument** or the larger number on the right of the base.
4. Finally, the number on the right of the equal sign is the **exponent**.

For instance, in $\log_3 9 = 2$, the base is 3, the argument is 9, and the exponent is 2.

The logarithm shows us what exponent must be used so that the base will be equal to the argument. In the case of $\log_3 9 = 2$, it tells us that 2 must be used as the exponent so that 3 becomes 9.

Here is the general notation for the logarithms:

$$\log_a m = n$$

Where a, m, and n are real numbers with $a \neq 1$ and $m \neq 0$. a is called the base, m is the argument, and n is the exponent.

Note: *The "formal" way to read* $\log_b x = y$ *is "logarithm of x to the base b is equal to y."*

Evaluating Logarithms

In this section, we will discuss how to evaluate a logarithm. Evaluating a logarithm means identifying an exponent that must be used for the base so that we can obtain the argument.

Let us compute for $\log_4 16$. This means we must think of the exponent used for 4 to get a value of 16.

Note that if we multiply 4 by itself, we can get 16 (i.e., 4 x 4 = 16); hence, we must use an exponent of 2 for 4 to get 16.

Therefore, $\log_4 16 = 2$.

Sample Problem: Compute the value of the following:

1. $\log_3 27$
2. $\log_2 32$
3. $\log_5 25$
4. $\log_7 1$

Solution:

1. Note that if we multiply 3 by itself three times, we can get 27 (i.e., 3 x 3 x 3 = 27). Therefore, $\log_3 27 = 3$.
2. If we multiply 2 by itself five times, we can get 32 (2 x 2 x 2 x 2 x 2 = 32). Therefore, $\log_2 32 = 5$.
3. Multiplying 5 by itself two times will result in 25. Therefore, $\log_5 25 = 2$.

4. To evaluate the value of $\log_7 1$, we think of a power to which we must raise 7 so that we obtain 1. By the zero-exponent rule, we know that any real number raised to zero is equal to 1. Hence, we must raise 7 to the power of 0 to get 1. Therefore, $\log_7 1 = 0$.

Evaluating Logarithms With an Argument of 1

Recall that in general, logarithms are expressed as $\log_a m = n$ where *a*, *m*, *n* are real numbers with $a \neq 1$ and $m \neq 0$. But what if the argument (or *m*) is equal to 1? How can we evaluate the value of the logarithm?

Say we want to evaluate the value of $\log_6 1$.

$\log_6 1$ tells us what exponent we should raise 6 so that we can get 1.

Recall that the zero-exponent rule states that any real number raised to 0 will result in 1. So, if we raise 6 to the power of 0, we can obtain 1. Mathematically, $6^0 = 1$.

Therefore, the value of $\log_6 1$ must be equal to 0 ($\log_6 1 = 0$).

To sum up, if the argument of the logarithm is 1, then the value of the logarithm is automatically 0 as per the zero-exponent rule:

$\log_a 1 = 0$, where a is a real number

Transforming Exponential Form Into Logarithmic Form and Vice Versa

1. Exponential Form Into Logarithmic Form

To convert a quantity in exponential form into logarithmic form, follow the steps below:

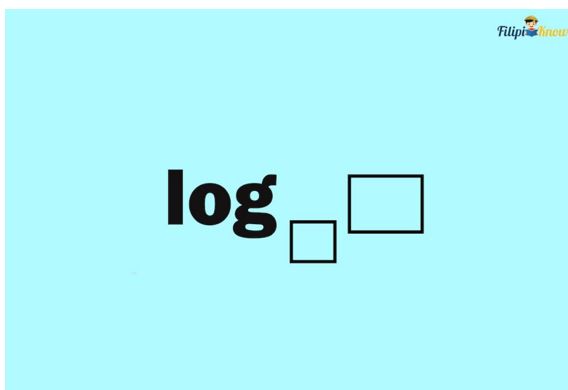
1. Write the “log” sign to indicate that you’re using the logarithm operator
2. Write the base of the exponential form as the base of the logarithmic form (i.e., the small number on the right of the log)
3. Write the value of the exponential form as the argument of the logarithmic form (i.e., the number on the right of the base of the logarithmic form)
4. Write the exponential form as the exponent of the logarithmic form

In general, a quantity in exponential form $b^y = x$ is written as $\log_b x = y$ in logarithmic form.

Sample Problem: Write $5^3 = 125$ into logarithmic form.

Solution: Using the steps on transforming the exponential form into the logarithmic form:

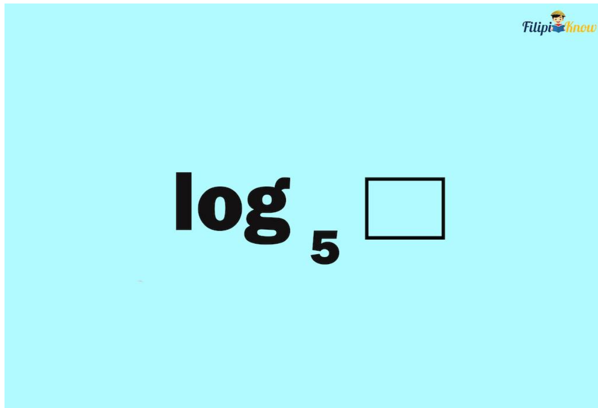
Step 1: *Write the “log” sign to indicate that you’re using the logarithm operator*



Note: the boxes in the expression above serve as placeholders for the remaining components of the logarithm.

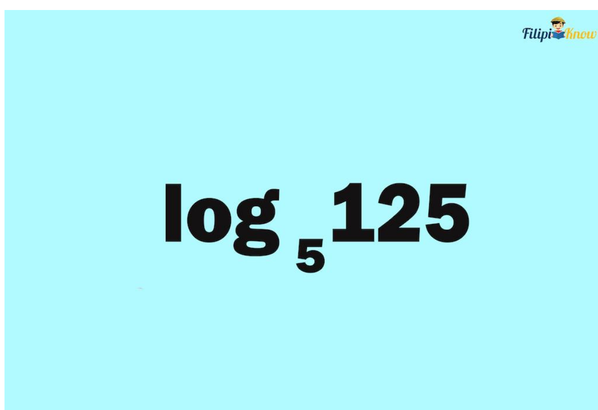
Step 2: *Write the base of the exponential form as the base of the logarithmic form (i.e., the small number on the right of the log)*

The base of $5^3 = 125$ is 5. Hence we use it as the base of the logarithmic form:



Step 3: *Write the value of the exponential form as the argument of the logarithmic form (i.e., the number on the right of the base of the logarithmic form)*

The value of the exponential form $5^3 = 125$ is 125. Hence, we will use it as the argument of the logarithmic form.



Step 4: *Write the exponential form as the exponent of the logarithmic form*

Finally, we use "3" (the exponent in $5^3 = 125$) as the exponent in the logarithmic form.

$\log_5 125 = 3$

2. Logarithmic Form Into Exponential Form

To transform a quantity expressed in logarithmic form into exponential form, we have to follow these steps:

1. Drop the “log” sign
2. Use the base of the logarithmic form (i.e., the small number on the right of the “log” symbol) as the base of the exponential form
3. Use the exponent of the logarithmic form (i.e., the number on the right of the equal sign) as the exponent for the exponential form
4. Use the argument as the value of the exponential form

Sample Problem: Express $\log_5 125 = 3$ into exponential form.

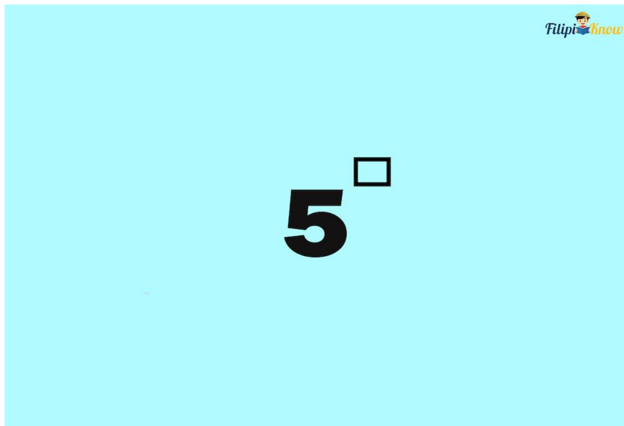
Solution:

Step 1: *Drop the “log” sign*

$$\cancel{\log}_5 125 = 3$$

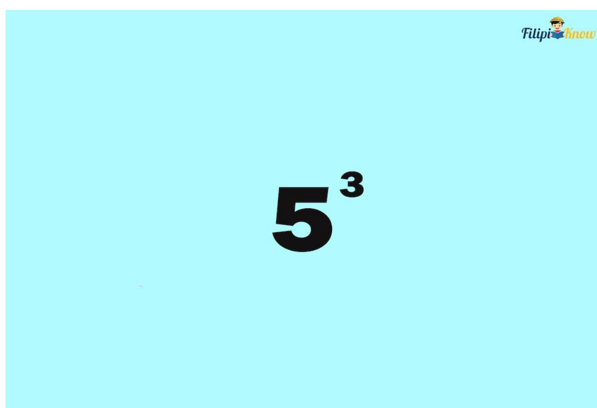
Step 2: *Use the base of the logarithmic form (i.e., the small number on the right of the “log” symbol) as the base of the exponential form*

The base of the logarithmic form is 5, so we use it as the base of the exponential form:



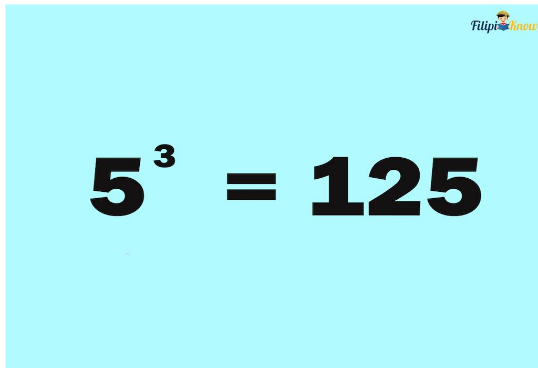
Step 3: *Use the exponent of the logarithmic form (i.e., the number on the right of the equal sign) as the exponent for the exponential form*

The exponent of the logarithmic form is 3 so we'll use it as the exponent in the exponential form.



Step 4: *Use the argument as the value of the exponential form*

Finally, we use 125 (the argument in $\log_5 125 = 3$) as the value of the exponential form.



Therefore, $\log_5 125 = 3$ can be written as $5^3 = 125$.

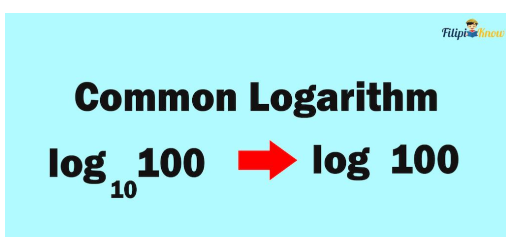
Common and Natural Logarithms

1. Common Logarithms

Common logarithms are logarithms with a base of 10.

For instance, $\log_{10} 100 = 2$ can be considered a common logarithm.

In a common logarithm, we don't need to write the "10" as the base anymore. We just leave it blank since it is already understood that we are dealing with a common logarithm.



For instance, log 100 = 2 automatically means $\log_{10}100 = 2$.

If we want to write a common logarithm to its equivalent exponential form, all we need to do is use 10 as the base, use the argument of the logarithm as the value of the exponential form, and then use the exponent of the logarithm form as the exponent of the exponential form. For instance, log 100 = 2 is equivalent to $10^2 = 100$.

$$\log 100 = 2 \rightarrow 10^2 = 100$$

Note that it's not always easy to compute the common logarithm of numbers. For instance, log 350 (which means $\log_{10}350$) cannot be determined easily through manual computation since its value is not a whole number ($\log_{10}350 = 2.544$). In these cases, scientific calculators are the most convenient tool to use. Since we are reviewing for exams that prohibit the use of any calculator, we will limit our discussion to common logarithms that can be evaluated through manual calculation only.

Sample Problem: Evaluate log 1000

Solution: log 1000 is a common logarithm. Although it appears that it has no base, it means that the base is 10. Therefore, log 1000 = $\log_{10}1000$.

To find the value of $\log_{10}1000$, we need to think of how many times 10 must be multiplied by itself to get 1000. Note that if we multiply 10 by itself thrice, the result will be 1000 (10 x 10 x 10 = 1000). Thus, $\log_{10}1000 = 3$.

2. Natural Logarithms

Natural logarithms use a “special number” as the base. This number is an irrational number represented as *e*. Formally, *e* is called Euler’s number (named after the mathematician Leonhard Euler). *e* is approximately equal to 2.718…

$\log_e 10$ is an example of a natural logarithm since it uses the base *e*.

However, we do not write natural logarithms like your ordinary logarithms. We use the symbol “ln” instead of “log,” and omit the base *e*. So, $\log_e 10$ can be written as ln 10 (read as *“natural logarithm of 10”*). Another example: $\log_e 1$ is equal to ln 1.

Natural Logarithm

$$\log_e 10 \rightarrow \ln 10$$

$$\log_e 1 \rightarrow \ln 1$$

If we write a natural logarithm to its equivalent exponential form, we just use the constant *e* as the base of the exponential form, use the argument as the value of the exponent, and use the value of the logarithm as the exponent of the exponential form. . For instance, if we transform ln x = y to exponential form, we’ll have $e^y = x$.

$$\ln x = y \rightarrow \log_e x = y \rightarrow e^y = x$$

Natural logarithms are widely applied in different real-life calculations such as for exponential growth and decay of microorganisms, compound interest, statistical analysis, calculus, physical sciences, and more.

Properties of Logarithms

Now that you are familiar with logarithms, let us now proceed to the discussion of their mathematical properties. These properties allow us to calculate operations involving logarithms.

Note that the properties below also apply to common logarithms and natural logarithms.

Property of Logarithms	In symbols
Product Property of Logarithms	$\log_a PQ = \log_a P + \log_a Q$
Quotient Property of Logarithms	$\log_a (P/Q) = \log_a P - \log_a Q$
Power Property of Logarithms	$\log_a P^q = q \log_a P$

1. Product Property of Logarithms

This property states that the logarithm of products can be expressed as the sum of the logarithms:

$$\mathbf{\log_a PQ = \log_a P + \log_a Q}$$

For instance, let us express $\log_2 8$ as a sum of logarithms. We can do this by simply thinking of some factors of 8.

4 and 2 are factors of 8 since 4 x 2 = 8. Therefore, we can express $\log_2 8$ as $\log_2(4 \times 2)$.

According to the product property, we can express $\log_2 8 = \log_2(4 \times 2)$ as the sum of the logarithms of the factors of 8: $\log_2 8 = \log_2(4 \times 2) = \log_2 4 + \log_2 2$.

Therefore, $\log_2 8 = \log_2 4 + \log_2 2$.

Sample Problem 1: Express $\log_2 5 + \log_2 4$ as a single logarithm.

Solution: By applying the product property of logarithms, we just simply multiply the arguments of the given logarithms.

$\log_2 5 + \log_2 4 = \log_2(5 \times 4) = \log_2 20$

Sample Problem 2: Given that $\log 3 \approx 0.48$ and $\log 2 \approx 0.30$. What is the approximate value of $\log 6$?

Solution: Since log 3 and log 2 are both common logarithms, they have the same base which is 10. This means that we can apply the product property to them.

Take note that "$\log 3 \approx 0.48$" means that the approximate value of log 3 is 0.48. This means that the value of log 3 is not exactly 0.48 since this value is just an approximation. The " ≈ " symbol denotes an approximation of a particular quantity.

Applying the product property, we have: $\log 6 = \log (3 \times 2) = \log 3 + \log 2$

This means that the value of log 6 is equal to the value of log 3 + log 2:

$\log 6 = \log 3 + \log 2$

Using the given approximate values of log 3 and log 2:

$\log 6 = \log 3 + \log 2$

$\log 6 \approx 0.48 + 0.30$

$\log 6 \approx 0.78$

Hence, the approximate value of log 6 is 0.78 (the value of log 6 when computed using a calculator is 0.77815… which is extremely near to our obtained value).

Sample Problem 3: Expand $\log_4(7a(b + 4))$ using the product property of logarithms.

Solution: Note that in $\log_4(7a(b + 4))$, the argument 7a(b + 4) is the product of 7a and b + 4. Furthermore, 7a is also the product of two quantities: 7 and a.

Therefore, $\log_4(7a(b + 4)) = \log_4(7 \cdot a \cdot (b + 4))$. By product rule, we can now express this as the sum of logarithms:

$\log_4(7a(b + 4)) = \log_4(7 \cdot a \cdot (b + 4)) = \log_4 7 + \log_4 a + \log_4(b + 4)$

2. Quotient Property of Logarithms

This property states that the logarithm of the quotient can be expressed as the difference of logarithms.

$$\log_a\left(\frac{P}{Q}\right) = \log_a P - \log_a Q$$

For instance, let us express $\log_2 6$ as the difference between logarithms. We just think of two numbers such that when these numbers are divided, the result will be equal to 6. For instance, 12 ÷ 2 = 6. Hence, $\log_2 6$ can be expressed as $\log_2(12⁄2)$.

By the quotient property of logarithms, we can express $\log_2 6$ as the difference between the logarithm of the numbers that when divided results in 6: $\log_2(12⁄2) = \log_2 12 - \log_2 2$.

Sample Problem 1:

$$\text{Expand } \log\left(\frac{x-1}{3}\right)$$

Solution: The argument of is

$$\frac{x-1}{3}$$

which is the quotient of x - 1 and 3. By the quotient property of logarithms, we can express the logarithm of the quotient of x - 1 and 3 as the difference between the logarithm of x - 1 and the logarithm of 3.

By applying the quotient property of logarithms:

$$log\left(\frac{x-1}{3}\right) = log(x - 1) - log\,3$$

Sample Problem 2:

Expand $\log\left(\frac{9}{b}\right)$

Solution: The argument is the quotient of 9 and b. By the quotient property of logarithms, we can express the logarithm of a quotient of 9 and b as the difference between the logarithm of 9 and the logarithm of b.

By applying the quotient property of logarithms:

$$log\left(\frac{9}{b}\right) = log\,9 - log\,b$$

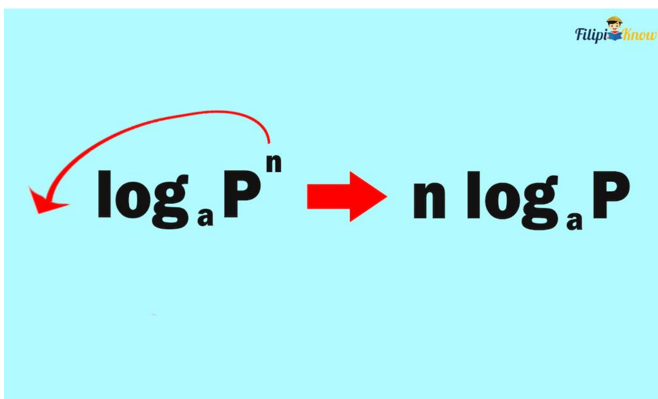
3. Power Property of Logarithms

The power rule of logarithms states that the logarithm of a quantity raised to an exponent is equal to the exponent times the logarithm of the quantity.

$$\log_a P^q = q \cdot \log_a P$$

For instance, $\log_2 x^2$ is a logarithm with a quantity raised to an exponent (x^2). By the power property of logarithms, we can express it as follows: $2 \cdot \log_2 x$

Basically, the power property of logarithms allows you to "move" the exponent to the left of the logarithm symbol.



Sample Problem 1: Expand $\log a^3$

Solution: Since we are dealing with a logarithm that has an argument with a quantity raised to an exponent (a^3), we can apply the power property of logarithms.

$\log a^3 = 3 \cdot \log a = 3 \log a$

Hence, the answer is 3 log a.

Sample Problem 2: Expand $\ln j^2$

Solution: By applying the product property of logarithms:

$\ln j^2 = 2 \cdot \ln j = 2 \ln j$

Therefore, the answer is 2 ln j.

More Sample Problems on Properties of Logarithms

Sample Problem 1: Expand $\log_4 a^2 b$

Solution: The argument in the expression $\log_4 a^2 b$ is $a^2 b$ which is the product of a^2 and b. Hence, we can apply the **product property of logarithms** to start expanding the given expression:

$\log_4 a^2 b = \log_4 a^2 + \log_4 b$

Note that we can still expand the expression $\log_4 a^2$ since we have an argument with a quantity raised to an exponent (a^2). We can apply the **power property of logarithms** for this case:

$\log_4 a^2 b = \log_4 a^2 + \log_4 b$

$$\log_4 a^2 b = 2 \log_4 a + \log_4 b$$

Hence, the expanded form of $\log_4 a^2 b$ is $2 \log_4 a + \log_4 b$.

Here's the summary of what we have performed above:

$\log_4 a^2 b$

$\log_4 a^2 + \log_4 b$ *Product Property of Logarithms*

$2 \log_4 a + \log_4 b$ *Power Property of Logarithms*

Sample Problem 2: Expand ln x^2y^3

Solution: The argument in the expression ln x^2y^3 is x^2y^3 or x^2 and y^3. This means we can apply the **product property of logarithms** to expand the given expression.

ln x^2y^3 = ln x^2 + ln y^3

Note that we can still expand both ln x^2 and ln y^3 since they involve arguments that are quantities raised to an exponent (x^2 and y^3). Thus, we can apply the **power property of logarithms** for this case:

ln x^2y^3 = ln x^2 + ln y^3

ln x^2y^3 = 2 ln x + 3 ln y

Hence, the answer for this example is 2 ln x + 3 ln y.

Here's the summary of our calculation above:

ln x^2y^3

ln x^2 + ln y^3 *Product Property of Logarithms*

2 ln x + 3 ln y *Power Property of Logarithms*

Sample Problem 3:

$$\text{Expand } log\left(\frac{3a^2}{2(a+b)}\right)$$

Solution:

The argument of the given logarithm is the quotient of $3a^2$ and 2(a + b). Hence, we can apply the quotient property to expand the given logarithm.

$$log\ (\frac{3a^2}{2(a+b)})\ =\ log\ 3a^2\ -\ log\ 2(a\ +\ b)$$

Notice that we can expand both log $3a^2$ and log $2(a + b)$. Let us expand log $3a^2$ first.

The argument of log $3a^2$ is $3a^2$ which is the product between 3 and a^2. By product property:

log $3a^2$ = log 3 + log a^2

We can still expand log a^2 by applying the power property:

log $3a^2$ = log 3 + 2 log a

Therefore, we now have:

$$log\ (\frac{3a^2}{2(a+b)})\ =\ log\ 3a^2\ -\ log\ 2(a\ +\ b)$$

$$log\ (\frac{3a^2}{2(a+b)})\ =\ (log\ 3\ +\ 2\ log\ a)\ -\ log\ 2(a\ +\ b)$$

Now, let us focus on log 2(a + b). The argument here is 2(a + b) which is the product of 2 and a + b. Hence, we can apply the product property:

$$log\ (\frac{3a^2}{2(a+b)})\ =\ (log\ 3\ +\ 2\ log\ a)\ -\ log\ 2(a\ +\ b)$$

$$log\ (\frac{3a^2}{2(a+b)})\ =\ (log\ 3\ +\ 2\ log\ a)\ -\ (\ log\ 2\ +\ log(a\ +\ b))$$

Thus, the final answer is (log 3 + 2 log a) - (log 2 + log(a + b))

Here's the summary of what we have performed above:

$\log\left(\frac{3a^2}{2(a+b)}\right)$

$\log 3a^2 - \log 2(a + b)$	*Quotient Property of Logarithms*
$(\log 3 + \log a^2) - \log 2(a + b)$	*Product Property of Logarithms*
$(\log 3 + 2 \log a) - \log 2(a + b)$	*Power Property of Logarithms*
$(\log 3 + 2 \log a) - (\log 2 + \log(a + b))$	*Product Property of logarithms*

Simplifying Logarithms

Simplifying logarithmic expressions means expressing them as a single logarithm. We can do this by applying the properties of logarithms we've learned from the previous section.

For instance, suppose we want to express $\log_3 10 + \log_3 5$ as a single logarithm.

The expression $\log_3 10 + \log_3 5$ involves the mathematical operation of addition. Note that we have learned in the previous section that the logarithm of a product can be expressed as the sum of logarithms (product property). Hence, we can apply the reverse of this property and express the sum of logarithms as the logarithms of a product.

By product property, we can express $\log_3 10 + \log_3 5$ as the logarithm of the product of 10 and 5 (to the base 3):

$\log_3 10 + \log_3 5$

$\log_3(10 \times 5) = \log_3 50$ *Product Property of Logarithms*

Hence, the simplified form of $\log_3 10 + \log_3 5$ is $\log_3 50$.

Sample Problem 1: Simplify $\log_2 8 + \log_2 7$

Solution: By the product property, we can express the sum of logarithms as the logarithm of their product:

$\log_2 8 + \log_2 7$

$\log_2(8 \times 7)$ *Product Property of Logarithms*

$\log_2 56$

Thus, the answer is $\log_2 56$.

Sample Problem 2: Simplify $\log_3(x + 5) + \log_3 x + \log_3 4$

Solution: By the product property of logarithms, we can express the sum of logarithms as the logarithm of their product:

$\log_3(x + 5) + \log_3 x + \log_3 4$

$\log_3[(x + 5) \cdot x \cdot 4]$

$\log_3(4(x)(x + 5))$

$\log_3(4x(x + 5))$

$\log_3(4x^2 + 20x)$

Thus, the simplified form is $\log_3(4x^2 + 20x)$.

Sample Problem 3: Apply the properties of logarithms to simplify ln 6 + ln 3 - ln 2

Solution: We are now dealing with two operations in the given logarithmic expression: addition and subtraction.

Let us deal with the addition sign first. By product property, we can express the sum of logarithms as the logarithm of their product:

ln 6 + ln 3 - ln 2

ln (6 x 3) - ln 2

ln 18 - ln 2

The resulting logarithmic expression ln 18 - ln 2 can be simplified further. The difference between logarithms can be expressed as the logarithm of their quotient using the quotient property. Thus:

ln 18 - ln 2 = ln (18⁄2) = ln 9

Thus, the final answer is ln 9.

Here's a summary of our solution above:

ln 6 + ln 3 - ln 2

ln (6 x 3) - ln 2 *Product Property of Logarithms*

ln 18 - ln 2

ln (18/2) *Quotient Property of Logarithms*

ln 9

Sample Problem 4: Simplify 3 $\log_2 a$ + $\log_2 b$

Solution: Note that by power rule, we can express the product of a constant and the logarithm of a quantity as the logarithm of the quantity raised to the constant. Note that 3 $\log_2 a$ is the product of a constant (which is 3) and a logarithm of a quantity ($\log_2 a$). Hence, we can apply the power rule to simplify it:

3 $\log_2 a$ + $\log_2 b$

$\log_2 a^3$ + $\log_2 b$ *Power Rule of Logarithms*

Now, we can simply apply the product rule to complete the solution:

$\log_2 a^3 + \log_2 b$

$\log_2 a^3 b$ *Product Rule of Logarithms*

Thus, the final answer is $\log_2 a^3 b$.