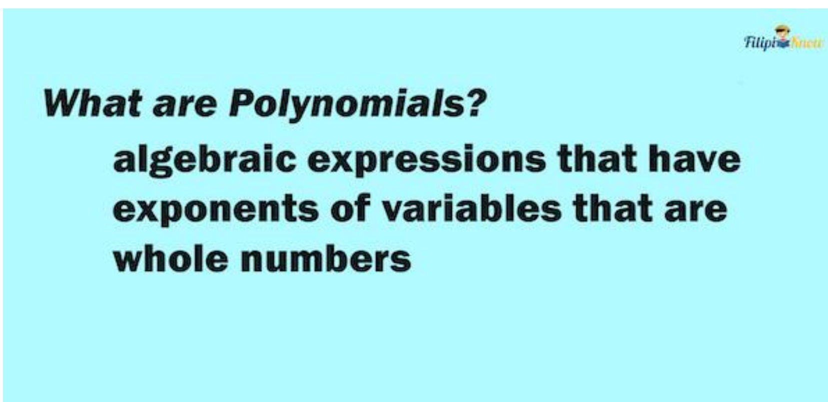
Algebraic expressions come in different forms. However, there are certain expressions in algebra that are really important and offer a lot of practical applications. These expressions are called polynomials.

In this reviewer, we'll discuss what polynomials are, the types of polynomials, and how to perform mathematical operations with them.

What are polynomials?

A polynomial is an algebraic expression where the exponents of its variables are whole numbers.



As you may recall, whole numbers are positive counting numbers including 0. Thus, the exponents in the variables of a polynomial are all positive counting numbers or 0. If a variable has an exponent of 0, this means that the variable is a constant.

$3x^4 + 2x^3 - x^2y + 3$ is an example of a polynomial since all of the exponents of the variables are whole numbers.

What expressions are NOT polynomials?

Now that you know what polynomials are, it is also important to know what makes an expression non-polynomial. Here's a list of algebraic expressions that are not considered polynomials:

1. Expressions with fractional or decimal exponents in the variable are not polynomials.

For example, $4x^{1/2} + 2y^3$ is not a polynomial since one of its variables, which is x, has a fractional exponent of ½.

2. Expressions with negative exponents in the variable are not polynomials.

For example, $2a^{-3}b - 5a^2b^3 + ab$ is not a polynomial since one of its variables, which is *a*, has a negative exponent which is - 3.

3. Expressions with variables in the denominator are not polynomials.

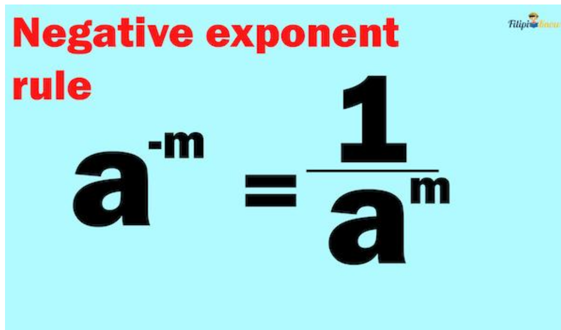
For example, 3x - 2⁄y is not a polynomial since it has a variable (which is y) in the denominator.

How about x + y⁄2? Is this a polynomial?

Although 2 is the denominator, 2 is not a variable. This means that x + y⁄2 can be considered as a polynomial since it has no variable in the denominator.

Why does a variable in the denominator disqualify an algebraic expression as a polynomial?

As per the negative exponent rule, if a variable is raised to a negative exponent, we should put that variable in the denominator so that the variable will now have a positive exponent.



If a variable is in the denominator, then it implies that before the negative exponent rule was applied, the variable had a negative exponent in the numerator.

We know that a negative exponent in the variable makes an expression a non-polynomial. This is the reason why variables in the denominator make an expression non-polynomial.

4. Expressions with variables under the radical sign are not polynomials.

Square root (√) and cube root (∛) are some of the examples of radical signs.

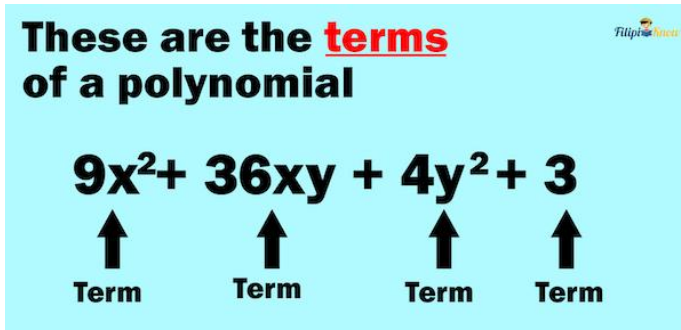
As an example, let's consider the expression √x - y. Since it has a variable (which is x) that is under the radical sign, then √x - y is not a polynomial.

How about √2 + x? Is this a polynomial?

Look at the radical sign. Note that 2 is inside the radical sign. 2 is a constant and not a variable. Thus, we can consider √2 + x as a polynomial.

Terms of a Polynomial.

A term in a polynomial consists of a number multiplied by a variable with a whole number exponent. The constant part is also a term of the polynomial.



The name *polynomial* comes from "*poly"* (Greek) which means many and *"nomen"* (Latin) which means name (in this case "term"). This means that a polynomial consists of different terms.

The terms in a polynomial are separated by addition or subtraction signs.

Take for example the polynomial $9x^2 + 36xy + 4y^2 + 3$. The terms in this polynomial are $9x^2$, $36xy$, $4y^2$, and 3. Notice that these terms are separated by addition signs.

Example: *Determine the terms in* $3x^2 + 5y - 2xz$

Solution: The terms in the given polynomials are $3x^2$, $5y$, and $2xz$.

Like Terms.

Two or more terms are like terms if their variables and exponents (of the variables) are the same. The numerical coefficient of like terms can be different.

For example, $5y^2$ and $3y^2$ are like terms because these terms have the same variable (*y*) and their exponents are the same (which is 2).

On the other hand, $7x^2$ and $5a^2$ are not like terms because these terms have different variables.

Also, $5x^2$ and $5x^3$ are not like terms because even if the variables of these terms are the same, the exponents are different.

Example 1: *Are 2xz and 5xz like terms?*

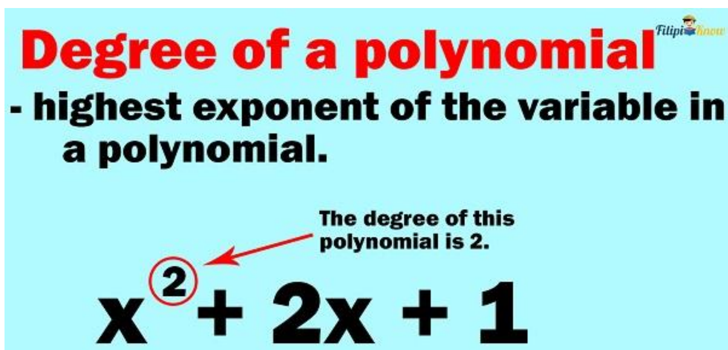
Solution: Yes, because these terms have the same variables (which are *x* and *z*).

Example 2: *Which of the following does not belong to the group of like terms: $5a^2b$, $-4a^2b$, $3a^2b^2$, and $9a^2b$.*

Solution: $3a^2b^2$ does not belong to the group because it has a different exponent for its variable b.

Degree of a Polynomial.

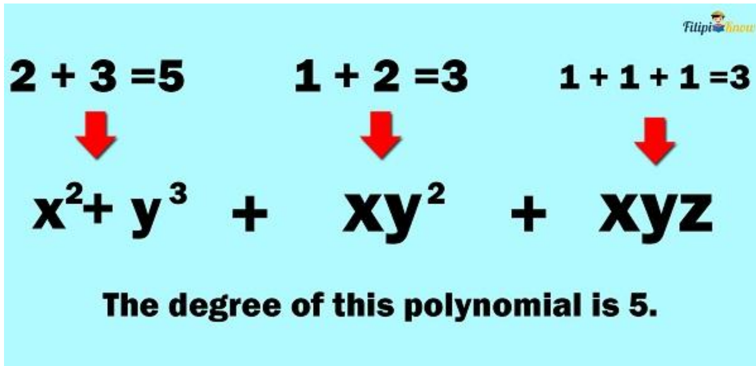
The degree of a polynomial is the highest exponent of the variable of a polynomial.



Example: *What is the degree of the polynomial $3k^7 - 5k^2 + k - 9$?*

Solution: The highest exponent of the variable in the polynomial is 7. Thus, the degree of the polynomial is 7.

If a polynomial has more than one variable, we add the exponents on each term and the highest resulting sum is the degree of the polynomial.



Types of Polynomials.

We can classify polynomials according to the number of terms they have or according to their degree.

1. Types of polynomials according to the number of terms.

Types of Polynomials According to Number of Terms	Number of Terms
Monomial	A polynomial with 1 term
Binomial	A polynomial with 2 terms
Trinomial	A polynomial with 3 terms
Multinomial	A polynomial with 4 terms and above

If we are going to base the classification of a polynomial according to the number of terms, the polynomial could be a monomial, binomial, trinomial, or multinomial.

- **Monomial** - an expression that has one term. This means that a constant, a variable, or a product of a constant and a variable with an exponent is a monomial. Examples: 4, a, 5x, $-9y^2$, and $4a^2b$.
- **Binomial** - an expression that has two terms. Examples: x + 2, 2y + z, $4ab - 3b^2$, and $p^2q - 3$.
- **Trinomial** - an expression that has three terms. Examples: $x^2 + 2xy + y^2$, 3x - y + 5, and $6x^3y + y - 9$.
- **Multinomial** - an expression with more than three terms. Examples: a + 2ab + 3abc + 4bcd and $4x^2yz + xy^3z - xyz^2 + xyz + 1$.

2. Types of polynomials according to degree.

Types of Polynomials According to Degree	Degree
Constant	0
Linear	1
Quadratic	2
Cubic	3
Quartic	4
Quintic	5

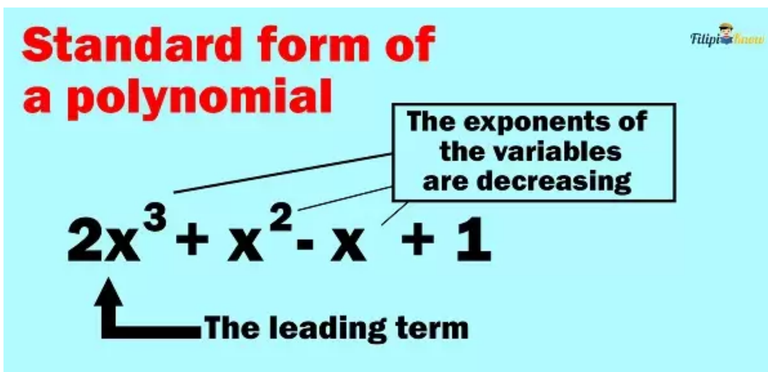
If we are going to base the classification of a polynomial according to its degree, the polynomial could be a constant, linear, quadratic, cubic, quartic, or quintic.

- **Constant** - a polynomial with a degree of 0. For example, the classification of 9 according to its degree is constant since the highest exponent it has is 0. Note that we can express 9 as $9x^0$.
- **Linear** - a polynomial with a degree of 1. For example, x + y is linear since the highest exponent of its variables is 1. Also, 3x + 9 is linear since its largest exponent of the variable is 1.
- **Quadratic** - a polynomial with a degree of 2. For example, $x^2 + 2x + 1$ is quadratic since the highest exponent of its variable is 2.

- **Cubic** - a polynomial with a degree of 3. For example, $x^3 + 3x^2 + 3x + 1$ is cubic since the highest exponent of its variable is 3.
- **Quartic** - a polynomial with a degree of 4. For example, $x^4 + 4x^2 + 8x + 2$ is quartic since the highest exponent of its variable is 4.
- **Quintic** - a polynomial with a degree of 5. For example, $x^5 + 5x^3 + 8x^2 + 2x - 3$ is quantic since the highest exponent of its variable is 5.

Standard Form of a Polynomial.

A polynomial is written in its standard form if the terms of the polynomial are arranged in a manner where the exponents of the variable are decreasing. In other words, the terms are written in a decreasing power of the variables.



If we express a polynomial in its standard form, the first term of that polynomial is called the leading term. Thus, the leading term is the term of the polynomial with the highest exponent.

Example 1: *Write $8y^2 + y^3 - 4y + y^5$ in standard form.*

Solution: We just write the terms of the polynomial in a manner such that the exponents of the variables are in decreasing order. Thus, the standard form of the given polynomial is $y^5 + y^3 + 8y^2 - 4y$.

Example 2: *What is the leading term of $5z^8 + 2z + z^7 - 2$?*

Solution: To determine the leading term of this polynomial, we should write it first in standard form. If we express the polynomial in standard form, we have $5z^8 + z^7 + 2z - 2$. The first term of this polynomial in standard form is $5z^8$. Therefore, $5z^8$ is the leading term.

Operations on Polynomials.

In this section, we are going to discuss how to perform the four fundamental operations on polynomials.

1. Addition of Polynomials.

To learn how to add polynomials, follow the steps below:

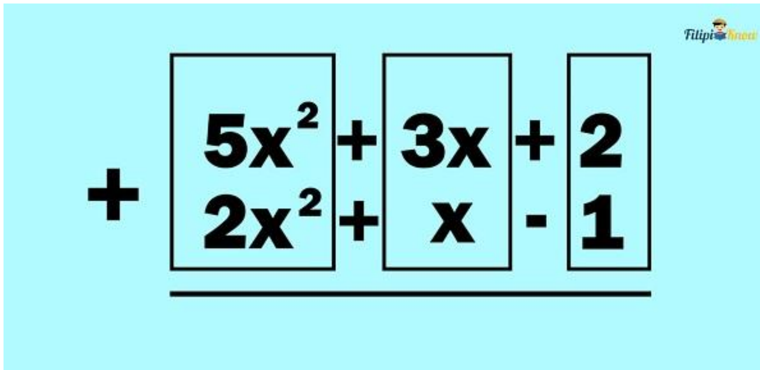
1. Arrange the given polynomials in standard form
2. Place the like terms of the given polynomials in columns
3. Add the like terms

Example 1: *Add* $5x^2 + 3x + 2$ *and* $2x^2 + x - 1$

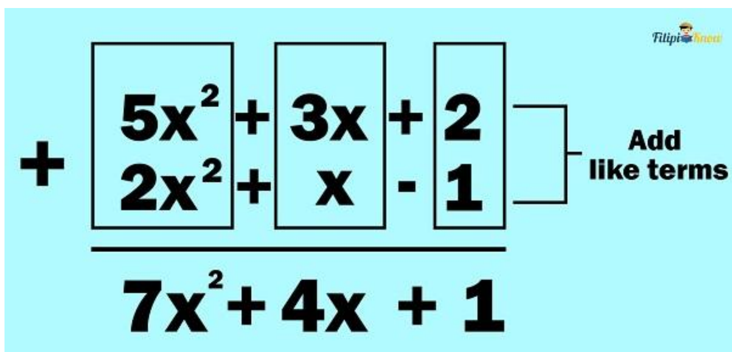
Solution:

Step 1: Arrange the given polynomials in standard form. Note that the given polynomials are already written in standard form since the exponents of their variables are in decreasing order. Hence, we can skip this step.

Step 2: Place like terms of the given polynomials in columns. Recall that the terms of given polynomials are like terms if the variables and the exponents of these variables are the same. Recall also that the numerical coefficients of like terms can be different.



Step 3: Add the like terms. To add like terms, we just add the numerical coefficients then copy the common variable and the exponent of it.



Thus, the answer is $7x^2 + 4x + 1$.

Example 2: *What is the sum if you add* $y^2 - 2z + x^3$, $5z - 3y + x^2$, and $2x^3 + 5y^2$?

Solution:

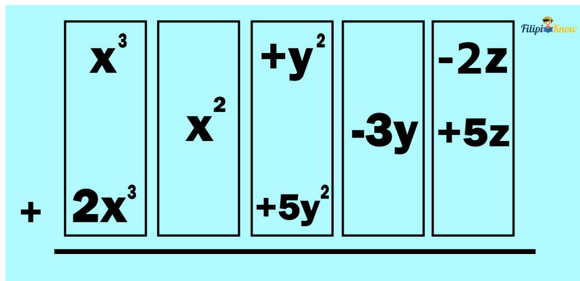
Step 1: Arrange the given polynomials in standard form. If we arrange the given polynomials into standard form, we will have the following:

$$x^3 + y^2 - 2z$$

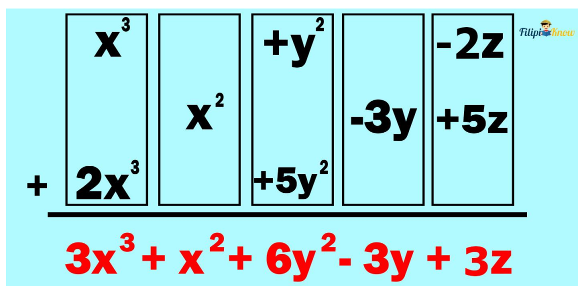
$$x^2 - 3y + 5z$$

$$2x^3 + 5y^2$$

Step 2: Place the like terms of the given polynomials in columns.



Step 3: Add the like terms.



Therefore, the sum is $3x^3 + x^2 + 6y^2 - 3y + 3z$

Example 3: *Add* $62xy - 5x^2y + 3$ *by* $-2x^2y + 10xy - y + 5$

Solution:

Step 1: Arrange the given polynomials in standard form. If we arrange the given polynomials into standard form, we will have the following:

$$-5x^2y + 62xy + 3$$

$$-2x^2y + 10xy - y + 5$$

Step 2: Place the like terms of the given polynomials in columns.

$$
\begin{array}{rrrr}
 -5x^2y & +\,62xy & & +\,3 \\
+\;\; -2x^2y & +\,10xy & -y & +\,5 \\
\hline
\end{array}
$$

Step 3: Add the like terms. Do not forget the rules on operations on integers when dealing with signed numbers.

$$
\begin{array}{rrrr}
 -5x^2y & +\,62xy & & +\,3 \\
+\;\; -2x^2y & +\,10xy & -y & +\,5 \\
\hline
 -7x^2y & +\,72xy & -\,y & +\,8
\end{array}
$$

The answer is $-7x^2y + 72xy - y + 8$.

2. Subtraction of Polynomials.

To know how to subtract polynomials, follow these steps:

1. Write the given polynomials in standard form
2. Change the sign into addition and reverse the sign of each term of the subtrahend (or the second polynomial)
3. Add the polynomials

Example 1: *Subtract $8x^2 - 2x + 1$ by $2x^2 - 5x - 7$*

Solution:

Step 1: Write the given polynomials in standard form. The given polynomials are already in standard form since their terms are already arranged based on the decreasing exponent of the variables.

Step 2: Change the sign into addition and reverse the sign of each term of the subtrahend (or the second polynomial). The subtrahend in the given problem is $2x^2 - 5x - 7$. If we reverse the sign of each term of this polynomial, we have $-2x^2 + 5x + 7$.

reversing the signs of each term

$$2x^2 - 5x - 7 \rightarrow -2x^2 + 5x + 7$$

Step 3: Add the polynomials. We are now going to add $8x^2 - 2x + 1$ to the polynomial we have obtained from step 2:

$$\begin{array}{rr} & 8x^2 - 2x + 1 \\ + & -2x^2 + 5x + 7 \\ \hline & 6x^2 + 3x + 8 \end{array}$$

Hence, the answer is $6x^2 + 3x + 8$.

Example 2: *What is the difference between $7x^5 - 5y + z^2$ and $5x^3 + 2x^5 - 6x - 2z^2$?*

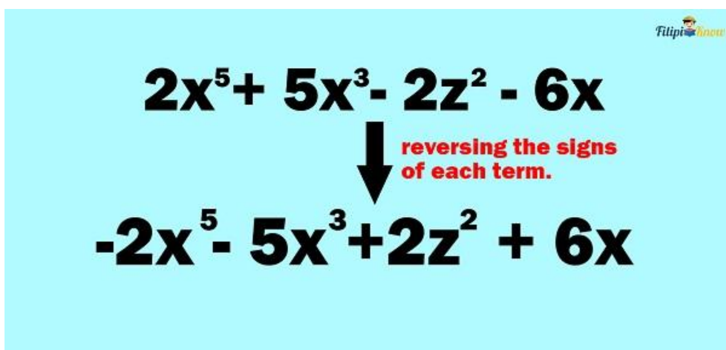
Solution:

Step 1: Write the given polynomials into standard form. If we arrange the given polynomials into standard form, we will obtain the following:

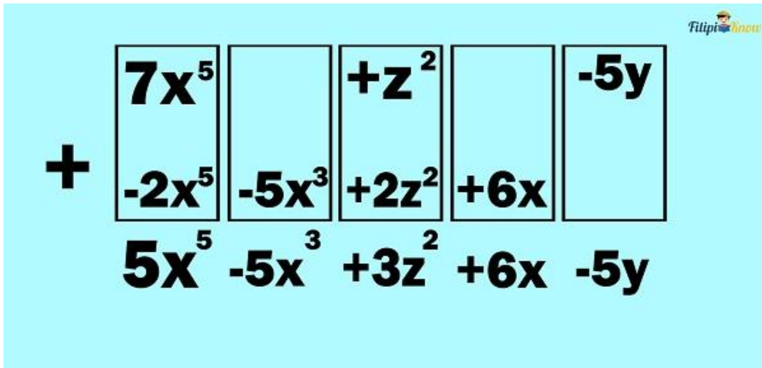
$$7x^5 + z^2 - 5y$$

$$2x^5 + 5x^3 - 2z^2 - 6x$$

Step 2: Change the sign into addition and reverse the sign of each term of the subtrahend (or the second polynomial). If we reverse the signs of each term of the second polynomial, we will obtain: $-2x^5 - 5x^3 + 2z^2 + 6x$



Step 3: Add the polynomials. We are now going to add $7x^5 + z^2 - 5y$ to the polynomial we have obtained from step 2:



Hence, the answer is $5x^5 - 5x^3 + 3z^2 + 6x - 5y$

3. Multiplication of Polynomials.

When we multiply polynomials, we apply the distributive property of multiplication over addition (or simply distributive property). Before we start discussing how to multiply polynomials, let's take a look first at what the distributive property is.

The distributive property tells us that multiplying the sum of two or more addends by a certain number is equal to the result when we multiply each addend by the same number.

In symbols,

$a(b + c) = ab + ac$

It looks like we distribute *a* to the addends *b* and *c*.

Example: *Compute for 4(5 + 9).*

Solution: If we use PEMDAS, we will obtain the following:

$$4(5 + 9)$$

$$4\ (14)$$

56

Thus, 4(5 + 9) = 56.

Now, let's try to apply the distributive property:

$$4(5 + 9) = 4(5) + 4(9) = 20 + 36 = 56$$

Thus, by applying the distributive property, we obtain the following: 4(5 + 9) = 56.

Keep in mind the concept of distributive property because we will apply this a lot when multiplying polynomials.

a. Multiplying a Polynomial by a Monomial.

Let us start with the simplest one: *How should we multiply a polynomial by a monomial?*

Suppose we have the polynomial $5x^2 + 3x - 1$ and we want to multiply it by a monomial like 2x.

Let us express our problem above as a mathematical sentence:

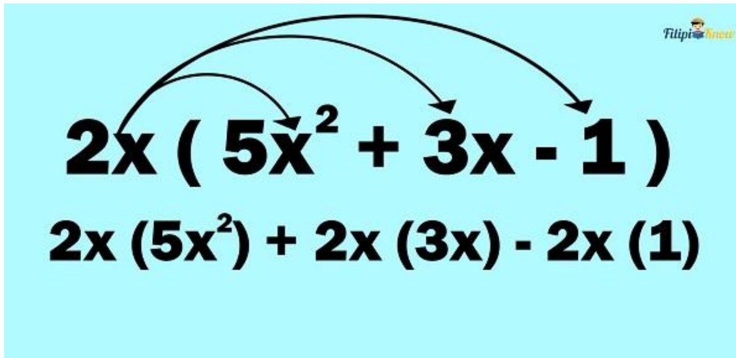
$$(5x^2 + 3x - 1)(2x)$$

Since multiplication is commutative (changing the position of numbers in a multiplication process will not change the result), we can express it as:

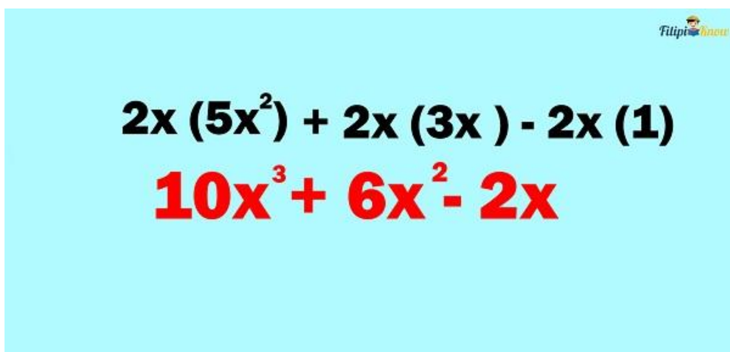
$$(2x)(5x^2 + 3x - 1)$$

Take a look at our mathematical sentence above. Notice that we are multiplying a certain quantity (2x) to a sum of addends $(5x^2 + 3x - 1)$. This means that we can apply the distributive property.

Applying the distributive property:



After we "distribute" 2x to the addends, we will perform multiplication. Take note that we apply the laws of exponents (the product rule, in particular) when we multiply the same variables.



Therefore, $(2x)(5x^2 + 3x - 1) = 10x^3 + 6x^2 - 2x$

Example: *Multiply 8ab + 2a - 3c by 4ab*

Solution: We have (4ab)(8ab + 2a - 3c). Applying the distributive property:

$$(4ab)(8ab) +(4ab)(2a) - (4ab)(3c)$$

Performing multiplication to each terms:

$$32a^2b^2 + 8a^2b - 12abc$$

Therefore, the answer is $32a^2b^2 + 8a^2b - 12abc$

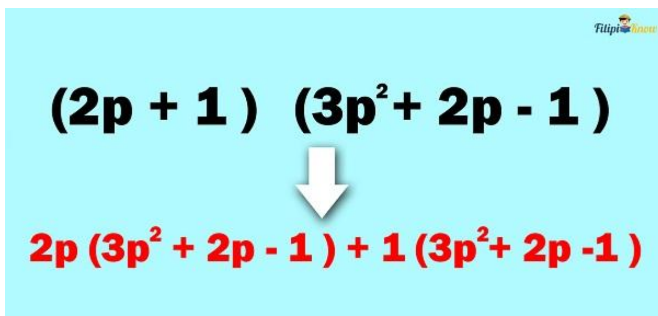
b. Multiplying a Polynomial by a Binomial.

Suppose we have the polynomial $3p^2 + 2p - 1$ and we want it to be multiplied by a binomial such as $2p + 1$. *How can we multiply these expressions?*

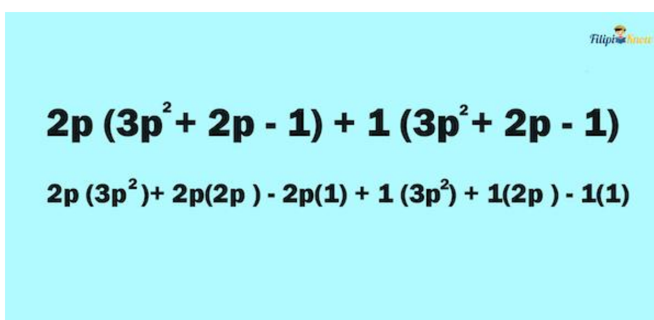
If we can express our given problem as a mathematical sentence, we have:

$$(2p + 1)(3p^2 + 2p - 1)$$

In this case, we can apply the distributive property. We can distribute the first term of the binomial which is 2p to $3p^2 + 2p - 1$ and we can also distribute the second term of the binomial which is 1 to the same polynomial (i.e., $3p^2 + 2p - 1$).



After we have distributed the terms of the binomial to the addends of the polynomial, we can apply again the distributive property:



Perform multiplication to each term and apply the product rule.

$$2p\,(3p^2 + 2p - 1) + 1\,(3p^2 + 2p - 1)$$

$$2p\,(3p^2) + 2p(2p) - 2p(1) + 1\,(3p^2) + 1(2p) - 1(1)$$

$$6p^3 \quad + 4p^2 \quad - 2p \quad + \quad 3p^2 \quad + \quad 2p \quad - 1$$

Note that we can combine some of the like terms of the resulting polynomial:

$$6p^3 + 4p^2 - 2p + 3p^2 + 2p - 1$$

$$= 6p^3 + 7p^2 - 1$$

Therefore, the answer is $6p^3 + 7p^2 - 1$

Example: *Multiply $5a^2 - 3ab + 2$ by $a - 2b$*

Solution: We have: $(a - 2b)(5a^2 - 3ab + 2)$. Distributing each terms of the binomial to the polynomial:

$$a(5a^2 - 3ab + 2) - 2b(5a^2 - 3ab + 2)$$

Applying the distributive property:

$$[a(5a^2) - a(3ab) + a(2)] - [2b(5a^2) - 2b(3ab) + 2b(2)]$$

Multiplying each terms:

$$[5a^3 - 3a^2b + 2a] - [10a^2b - 6ab^2 + 4b]$$

We can rewrite the expression above as:

$$[5a^3 - 3a^2b + 2a] + [-10a^2b + 6ab^2 - 4b]$$

Combining like terms:

$$5a^3 - 13a^2b + 6ab^2 + 2a - 4b$$

Therefore, the answer is $5a^3 - 13a^2b - 6ab^2 + 2a - 4b$

c. Multiplying a Polynomial by Another Polynomial.

You have learned how to multiply a polynomial by a monomial or a binomial in the previous sections of this reviewer. Recall that we just applied the distributive property to multiply these expressions. Using the same technique, you can also multiply a polynomial by another polynomial.

To multiply a polynomial by another polynomial, simply multiply each term of one polynomial by each term of the other polynomial and then combine like terms of the resulting polynomial.

Example 1: *Multiply $3a^2 + 2a - 1$ by $a^3 - 4a + 1$.*

Solution: Expressing the mathematical problem above as a mathematical sentence, we have:

$$(3a^2 + 2a - 1)(a^3 - 4a + 1)$$

We then multiply each term of $3a^2 + 2a - 1$ to $a^3 - 4a + 1$

$$3a^2 + 2a - 1\,(a^3 - 4a + 1)$$

$$\downarrow$$

$$3a^2\,(a^3 - 4a + 1) + 2a\,(a^3 - 4a + 1)$$
$$-\ 1\,(a^3 - 4a + 1)$$

Applying the principle of distributive property:

$$3a^2(a^3) - 3a^2(4a) + 3a^2(1) + 2a(a^3)$$
$$-2a\,(4a) + 2a\,(1) - 1\,(a^3)$$
$$(-4a) - 1\,(1)$$

Putting together the terms and combining like terms:

$$3a^5 - 12a^3 + 3a^2 + 2a^4 - 8a^2 + 2a - a^3 + 4a - 1$$
$$= 3a^5 + 2a^4 - 13a^3 - 5a^2 + 6a - 1$$

Therefore, the answer is $3a^5 + 2a^4 - 13a^3 - 5a^2 + 6a - 1$.

Example 2: *Multiply* $5x^2y + y + 2$ *by* $y^2 + 3x - 1$

Solution: Expressing the mathematical problem above as a mathematical sentence, we have:

$$(5x^2y + y + 2)(y^2 + 3x - 1)$$

We multiply each term of $5x^2y + y + 2$ to $y^2 + 3x - 1$:

$$5x^2y(y^2 + 3x - 1) + y(y^2 + 3x - 1) + 2(y^2 + 3x - 1)$$

Applying the distributive property:

$$5x^2y(y^2 + 3x - 1) + y(y^2 + 3x - 1) + 2(y^2 + 3x - 1)$$

$$(5x^2y^3 + 15x^3y - 5x^2y) + (y^3 + 3xy - y) + (2y^2 + 6x - 2)$$

Combining like terms and writing the resulting polynomial in standard form:

$$(5x^2y^3 + 15x^3y - 5x^2y) + (y^3 + 3xy - y) + (2y^2 + 6x - 2)$$

$$5x^2y^3 + 15x^3y - 5x^2y + y^3 + 2y^2 + 3xy + 6x - y - 2$$

Therefore the answer is $5x^2y^3 + 15x^3y - 5x^2y + y^3 + 2y^2 + 3xy + 6x - y - 2$

In the next chapter, we are going to discuss different techniques to multiply polynomials under different conditions. We call these techniques “special products”. This includes multiplying a binomial by another binomial, squaring a binomial, cubing a binomial, and so on.

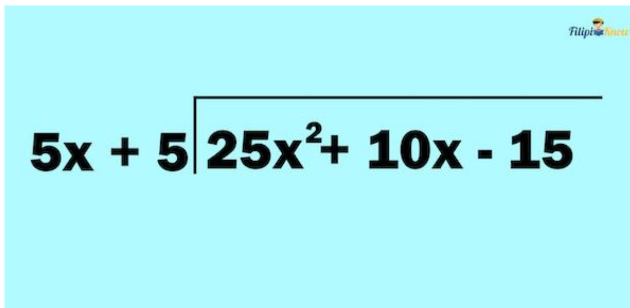
4. Division of Polynomials.

To divide polynomials, we use the long division method. This method allows us to use the division brackets which we also use when we divide whole numbers.

Suppose we want to divide $25x^2 + 10x - 15$ by $5x + 5$

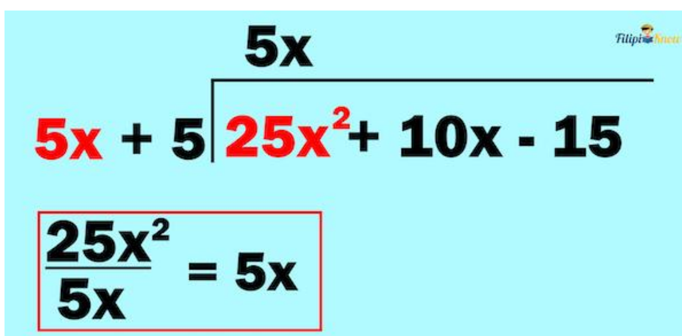
$25x^2 + 10x - 15 \div (5x + 5)$

The first thing we have to do is put $25x^2 + 10x - 15$ inside the division bracket and then put 5x + 5 outside of it.



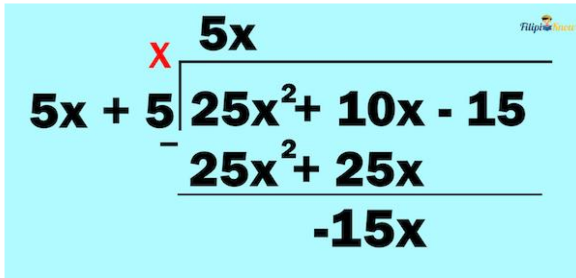
Next, we divide the first term of the polynomial inside the division bracket by the first term of the polynomial outside the division bracket. To perform this, we apply the quotient rule. We put the result above the bracket and align it to the first term of the polynomial inside the bracket.

We divide $25x^2$ by 5x and obtain 5x. We then put this answer (i.e., 5x) above the division bracket, making sure it's aligned to the first term of the dividend.

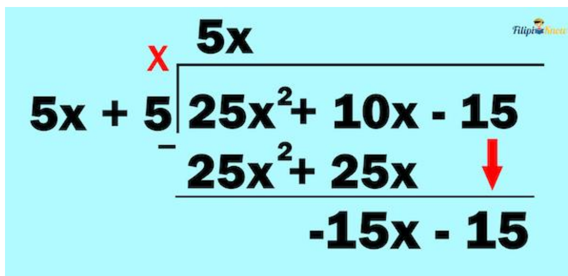


Then, we multiply the divisor by the answer we obtained earlier and subtract the answer from the dividend.

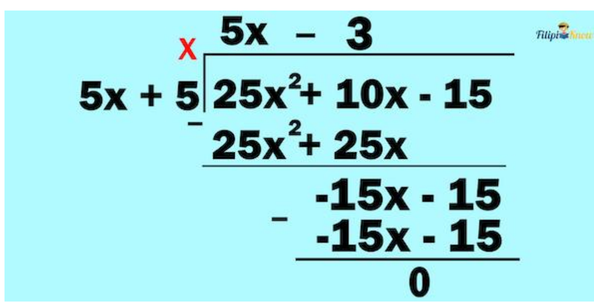
This means that we multiply 5x + 5 by 5x to obtain $25x^2 + 25x$. We then subtract $25x^2 + 25x$ from $25x^2 + 10x$ to obtain -15x.



We bring down - 15 to create a new polynomial.



We repeat the things we performed above.



Hence, the answer is 5x - 3.